Mathematical Tables
and other
Aids to Computation

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A Bell Telephone Laboratories' Computing Machine—II

This is the second of two articles by Dr. ALT describing a Bell Telephone Laboratories' machine. The first article, which appeared in the January issue of *MTAC*, treated the machine number storage, calculation, and input and output. Sections 2, 3, 4 of this first article are referred to in the second.

5. Control

By "control" is meant the process of causing the machine to perform the desired operations. This definition overlaps in part the concept of input discussed in the preceding section. The input consists of numbers and orders, and the input of orders is part of the control of the machine. In this section we shall consider the remaining aspects of the control, namely, what kinds of orders the machine can receive and how they are represented.

a. Calculator Orders.—The most frequently used orders are those which cause the machine to perform the ordinary arithmetic calculations of addition, subtraction, multiplication, division, and extraction of square root. Each of these orders contains the designations of two registers, the locations of the two numbers on which the operation is to be performed (in the case of square roots, only one such register), and of one further register, the location to which the result of the operation is to be transmitted. These locations are referred to as the first, second, and third "registrations." In the language of some of the more recent machine designers, they would be called "addresses."

Thus the control of this machine is accomplished, for the most part, by means of three-address orders. In contrast, the "Mark I" at Harvard[1] uses a two-address system. Of the projected electronic machines, one uses a four-address system and another a single-address system. In systems of less than three addresses several orders are necessary to accomplish arithmetic operations. For instance, for multiplication there would be one order stating that the number at location a should be used as a multiplicand. The next order states that the number at location b should be used as a multiplier. A third order designates location c to receive the product. In such a system, the orders are shorter and there are fewer types of orders, but more orders are required in any given program. It is not easy to see that any system is decidedly superior to the others.

In addition to the designation of locations, arithmetic orders for our machine contain an indication of the kind of operation to be performed, and indications as to whether the numbers stored in the first and second registrations are to be cleared (deleted) after the operation has been performed, or whether they are to be retained for future use.

The register to which the result is transferred must be clear at the time the operation is performed. If, erroneously, an attempt is made to transfer a number into a register which already contains a number, the machine will alarm. It will likewise alarm if one of the registers designated in the first two registrations of an order fails to contain a number at the time the operation starts.

As mentioned in section 4b, orders are expressed in a code whose elements are the first twenty letters of the alphabet; these are represented on

tape by combinations of three holes out of the possible six holes. Each order consists of several letters. (Occasionally, arithmetic symbols are used in place of letters. For instance, the code for addition is the letter P, but for convenience we sometimes write $+$ instead. The same three-hole code on the tape represents both symbols.)

Each letter or code has different meanings depending on its position within an order. For the purpose of coding and decoding, the code letters of an order are divided into groups in such a way that the meaning of a code letter is uniquely determined by the group in which it stands. For example, suppose that we want the machine to carry out a subtraction. Suppose that the minuend is located in register A (the fifteen storing registers of the machine are designated by the letters A through O), that the subtrahend is located in register C, that it is intended to store the difference in register B. Suppose also that the subtrahend will be needed for further computations and should therefore be held in register C, whereas the minuend is to be cleared after this subtraction. The order which accomplishes this operation is

$$AC - CH = B \, \overline{EO}.$$

The letters in this order belong to the following groups:

$$
\begin{array}{cccccccc}
A & C & - & C & H & = & B & \overline{EO} \\
1 & 2 & 3A & 4 & 5 & 5B & 6 & 7
\end{array}
$$

The groups are denoted by 1, 1A, 1B, 2, 2A, etc. The numbering is somewhat arbitrary and need not be explained in detail. Each order begins with group 1. The letters A to O in this group designate the corresponding storing registers and indicate that the number located in the designated register should be used as the first argument of the operation prescribed by the order. Letters P to T in this group have special meanings some of which will be mentioned later. In the above example, the letter A in group 1 indicates that the first argument of our operation (the minuend) is to be taken from register A.

The code in group 2 refers to clearing or holding the number in the register; the letter C, as in the above example, designates clearing, the letter H designates holding, and other letters have related meanings.

Group 3 is not used in this particular order (nor are groups 1A, 1B, 2A or 2B). The minus sign in group 3A indicates subtraction. (The letter O could have been used instead.)

Letters in groups 4 and 5 have a meaning corresponding to that of groups 1 and 2 but referring to the second argument. In our example, the C in group 4 indicates that the subtrahend is to be taken from register C, and the H in group 5 indicates that register C should not be cleared after this operation.

In group 5B the only code ever to appear is the sign $=$. It has no functional meaning but acts merely as an indicator of the fact that the first part of the order is completed. It enables the machine to overlap the reading of orders with calculation. It also serves as a check against operators' errors in coding, since the machine will stop if the equality sign does not appear at the proper place in the tapes.

In group 6, the letters from A to O designate the register to which the

result of the operation is to be transferred. The remaining letters are used for special purposes.

The symbol $\overline{EO}$ in group 7 denotes *the end of the order*. It is written here as two letters for typographical reasons, but on the tape it is represented by a single code. This code, like the equality sign mentioned before, has no functional meaning but serves to check and synchronize the operation of the machine. Every order, regardless of its contents, ends with the symbol $\overline{EO}$ or with one of two similar symbols which have the additional function of directing the control to the beginning of the next problem (or section of a problem) or back to the beginning of a repetitive cycle of computations.

All orders for arithmetic operations—addition, subtraction, multiplication, division, square root—are written in analogy to the above example, except that on square roots the second argument is omitted, e.g., $AC\sqrt{} = B\ \overline{EO}$ orders that the root of the number stored in register A be extracted, register A be cleared, and the root stored in register B.

Slight modifications of these orders, using some of the groups 1A, 1B, etc., which are not used in the above example, will have the effect of changing the sign or taking the absolute amount of one or both of the arguments, and of cancelling the rounding of the result or the shifting off of initial zeros (see section 3).

b. Transfer Orders.—Transfer orders accomplish the transfer of a number from one register to another. The first register may be cleared or held as desired. For example, the order $A\ C = B\ \overline{EO}$ results in transferring the number previously stored in register A into register B, and in clearing register A (as indicated by the letter C). It is necessary that previous to the receipt of this order register A contain some number and register B be clear; if either of these conditions is not met, the machine will give an alarm. The symbol $\equiv$ could have been replaced by the letter S.

c. Tape Orders.—On receipt of a tape order, the machine forms a four-digit page or block number and moves one of the table tapes forward or backward until the desired page or block is found. If this be a page number, nothing further happens until another tape order calls for a block. If a block has been called for, the machine, after finding the block, automatically transfers the first number in this block into the table register, where it is used (as described in 4f) as if it were in a storage register.

The tape order indicates which table tape is to be used, whether a page or block number is to be found, and in which way the page or block number is to be formed. Either of these numbers may be formed in one of four different ways: (1) by taking the first four digits of a number stored in any register; (2) by taking the first three digits of a number stored in one register, and the first digit of a number stored in another register, and using these four digits, in this order, as a four-digit block or page number; (3) by taking the first two digits from two numbers; or (4) by taking the first digit of one number and the first three of another. The last three cases are particularly useful in connection with functions of two variables, i.e., double-entry tables. The tape order also indicates whether the registers which furnish the page or block number are to be cleared or held.

d. Printing and Perforating.—The control of the printing of answers and of perforating numbers on the storage tape is effected by the second

part of a calculator or transfer order, rather than by a separate order.   For example, the order

$$A \ C - B \ H = Q \ P \ A \ G \ F \ \overline{EO}$$

indicates to the machine that the difference of the numbers in registers A and B is to be calculated and printed.   The five letters after the equality sign indicate whether the result is to be printed, perforated or both; whether the recording (printing or perforating) should be in the usual decimal notation or in one of the abbreviated notations which are used in the machine; and in the former case, whether the sign of the number is to be recorded or omitted, how many decimal places are to be recorded before and after the decimal point, and how many blank spaces should be left on the printing paper after the number. Other printing orders direct the perforating of page and block numbers on storage tape. Routine orders can only direct the printing of numbers. Alphabetical information is printed under the direction of a SWP (switch to printer) section of the problem tape.

e. Discrimination.—By discrimination is meant the process of selecting one of several possible courses of computation, depending on the outcome of a previous computation. In this machine there are two types of discrimination. The first and simpler type is similar to that used in most other machines. It is limited to two alternatives, between which the decision is made depending either (a) on the sign of a previously computed number or (b) on whether such a number is equal to or different from zero.   Possible alternatives are (a) continuing the computation with the next order on the routine tape, (b) moving to the beginning of the same or some other section of the routine tape, (c) going on to a new problem or to a new section of the same problem, (d) stopping and sounding an alarm.   For instance, a discrimination order might run like this: "Look at the number stored in register A. If the number is positive, go to section 2 of this routine tape; if it is zero or negative, go to section 4 instead." Or: "Subtract the numbers in registers A and B. If the difference is zero (within the seven-place accuracy of the machine), proceed with the next order; if it is different from zero, give the alarm."

Discrimination of this type is effected by the second part of a calculator on transfer order, rather than by a separate order. For instance, the order

$$A \ C \equiv P \ R \ B \ \overline{EO}$$

instructs the machine to proceed either to the next order or to the beginning of section 2, depending on whether the sign of the number stored in register A is − or +.

The machine is also equipped for a more complicated type of discrimination. In this, the sign of a number is transferred to a special organ of the machine, called the discriminator. The signs so transferred are counted, and their number is compared with a previously-set upper limit. For instance, suppose that the discriminator has been pre-set to count up to three plus signs and up to six minus signs. (The operator may choose any combination not exceeding four plus signs and seven minus signs.) An order might read as follows: "Take the number stored in register A and transfer its sign to the discriminator. If the sign is − and the number of minus signs trans-

ferred to the discriminator so far, including this one, is less than 6, proceed to the next routine order on the tape. If this is the sixth sign to be transferred to the discriminator, this problem is finished; go on to the start of the next problem. If the sign is $+$, and less than three plus signs have been transferred to the discriminator so far, go to section 3 of this routine tape. If this is the third plus sign, give the alarm." In this process, the count of signs will be released (set back to zero) under several conditions: (a) whenever the pre-set limit is reached, (b) the count of minus signs is released whenever a plus sign is recorded (i.e., only consecutive minus signs are counted), (c) the count of plus signs is released whenever a pre-set number of minus signs have been recorded since the last plus sign (i.e., sequences of plus signs need not be consecutive, but they must not be interrupted by more than a certain maximum number of minus signs). This limit for intervening minus signs may be chosen different from the limit for consecutive minus signs mentioned before, and must not exceed five.

This discrimination feature is rather complicated, and it would have been possible to achieve the same effect by a succession of simple orders rather than by building a special machine component. However, it is bought at a surprisingly low price. The discriminator contains only 71 relays.

One of the many possible uses of the count of signs in the discriminator is as follows. Suppose we wish to integrate a differential equation by a step-by-step method in which each step is iterated several times until the result of two successive steps agrees within a prescribed limit of accuracy. Suppose furthermore that we wish to adjust the length of the integration step in such a way as to reduce the computing time to a minimum. If too many iterations are needed at each step, it may be preferable to change to a shorter step; if in several successive steps the first iteration is correct, it may be economical to lengthen the step. To accomplish this, the machine can be instructed to compute, at the end of each iteration except the first, the absolute value of the difference between the results of the last and the preceding iteration. From this value, the desired limit of accuracy is then subtracted, and the sign of the difference is referred to the discriminator. A positive sign indicates that the desired accuracy has not yet been reached; a negative sign indicates that the result is sufficiently accurate and that the next integration step may be started. The instructions to the discriminator are: on receipt of a plus sign before the pre-set limit of such signs is reached, go back to the beginning of the routine tape (i.e., perform another iteration); on receipt of a minus sign before the limit is reached, go on to the next section of the problem (i.e., to the next integration step); on reaching the limit of plus signs (i.e., if too many iterations are needed for the desired accuracy), go to a section of the routine tape which will cause the length of the integration step to be decreased; on reaching the limit of minus signs (i.e., if for several successive steps the first iteration has given the desired accuracy), go to another section of the routine tape which will cause the length of the integration step to be increased.

f. Miscellaneous Orders.—It is hardly necessary to enumerate all the different types of orders which may be given to the machine. There are orders to clear a register, orders to the typewriter to feed a new line, orders to the discriminator to set up the desired limits for the count of signs,

orders to insure that the proper tapes have been loaded into all tape transmitters before beginning a problem, orders to decide whether, at the start of a new problem or of a new section of a problem, the numbers then in the machine should be held or cleared and whether numbers stored on the storage tape should be held available for future use or moved outside the machine.

There are in the machine eight "sign registers" capable of storing only a plus or minus sign. An order may be given to transfer from a storing register to a sign register, in which case the sign register will record the sign of the number previously held in the storage register. If that number was zero, a negative sign will appear in the sign register unless special provisions are made to the contrary. An order calling for transfer from a sign register to a storage register will cause the latter to be set to $+ 1$ or $- 1$. Also, one or both of the variables occurring in a calculator order may be taken from a sign register; the result is always the same as if one of the numbers $+ 1$ or $- 1$ had been used.

It is possible to replace one (but not both) of the variables in a calculator order by a constant which is put on the routine tape. This is indicated by the letter Q, followed by the desired constant. For instance, the order

$$\text{A H} \times \text{Q} + 4000000{+}01 \ \overline{\text{NC}} \ \text{C} = \text{B} \ \overline{\text{EO}}$$
$$\begin{array}{ccccccc} 1 & 2 & 3\text{A} & 4 & & 5 & 5\text{B} \ 6 & 7 \end{array}$$

in which the grouping of the code letters is indicated by the numbers written below them, causes the machine to multiply the number in register A by the constant factor 4 and put the product into register B. The number 4 is written in the form in which it will be stored in the machine; i.e., sign of number, seven digits, and "exponent." In ordinary notation this corresponds to $+ .4000000 \ 10^{+01}$. The symbol $\overline{\text{NC}}$ or *number check* indicates the end of the number.

The machine is equipped with a register called the "BTL Register," which, in addition to serving as a storing register, is capable of performing a number of special operations in connection with the forming of "block numbers" and with the computation of trigonometric and logarithmic functions. These operations usually involve some form of isolating and shifting of individual digits of a number. For instance, if a number is referred to the BTL register with orders to compute its antilogarithm, the register will automatically separate the characteristic from the mantissa, use the latter as the argument for computing the antilogarithm, and use the former for obtaining the exponent of the result. Similar operations are performed in computing the logarithm or a trigonometric function of a number. The operations involved in forming a page or block number, as described in section 5c, are also performed by means of the BTL register. (The letters "BTL" stand for block-forming, trigonometric, and logarithmic operations.) While these features were provided in the machine for their special purposes, they have been used by the operators of the machine for a variety of other purposes which involve manipulation of individual digits of a number.

The BTL register is closely associated with the so-called permanent function tables in the machine. These are wirings which make it possible to obtain very quickly the value of certain functions for any desired value of the argument. The functions so available include $\sin x$, $\cos x$, $\tan^{-1} x$, and on the machine of the Ordnance Department, $\log x$ and 10^x.

6. Operational Characteristics

a. Arrangement of Machine Components.—As was said before, two machines of the kind with which we are dealing are in existence at present. Each of these two installations is called a "computing system" and consists of two "computers" and associated equipment.

A computer contains all the components needed for dealing with a problem, with the exception of tape-reading devices (tape transmitters). Each computer contains:

Fifteen storage registers.

Eight sign registers.

A calculator, capable of performing addition, subtraction, multiplication, division and square root extraction.

A routine control, the organ which accepts orders from the routine tapes (and also from the SWR (switch to routine) sections of the problem tape), directs the carrying out of these orders by the computer, and directs the movement of the tapes from which the orders are received. It includes the routine register, an organ similar to a storing register, which receives numbers read from the routine tape (see section 5f above).

A BTL register (see section 5f above).

A table register and table control. The former receives and stores numbers read from table tapes, the latter directs the movement of the table tapes so as to find any desired location in them.

A problem register and problem control. The former receives and stores numbers read from CCS (computing cycle section) sections of the problem tape; the latter directs the movement of the problem tape (except in SWR sections) and directs the machine to act in accordance with various special symbols perforated on this tape.

A printer register and printer control. The former receives numbers from various machine components for the purpose of having them printed or perforated on the storage tape, and the latter directs the process of printing or perforating.

A discriminator (see section 5e above).

A recorder table, containing a printer (teletype page printer), a reperforator, a tape transmitter—the latter two for the storage tape—and a "distributor," which has the task of translating numbers from the relay representation used in the machine to the teletype code which is acceptable to the printer and reperforator.

In addition to the two computers, each installation contains the following:

Permanent tables, as described in section 5f above. These are accessible to both computers. The electrical connection between a computer and a permanent table is performed automatically under the direction of the table control.

Several (three or four) position tables.

Relay equipment connecting the position tables to the computers ("Position Connectors").

Auxiliary equipment, not electrically connected with the main machine.

The position tables are the input organs of the machine. Each of them holds twelve tape transmitters, one for the problem tape, five for routine tapes, and six for table tapes.

In general, a problem is solved on one computer, and the corresponding tapes are loaded into one position table. For large problems it is possible to use two computers and two position tables. In either case, each of the two computers is connected to one position table, and one or two position tables are unused. Into these spare positions the tapes for a problem can be loaded by the operating personnel while the machine is occupied with other problems. The "Position Connector" is so arranged that whenever one computer has finished the problem on which it has been working, it disconnects itself automatically from the position to which it had been connected, and connects itself to a loaded spare position. Thus the transition from one problem to the next is accomplished without any loss of machine time.

In addition to the very considerable saving in set-up time, setting problems up on spare position tables makes it possible to run the machine over night without anyone in attendance. The usual procedure is to load the machine up with problems in the evening and to come back in the morning to find the work done.

The auxiliary equipment consists of several hand perforators, on which the operating personnel produces the tapes to be used in the machine, and one or two "Tape Processors." These have the dual functions of translating tapes from the simple code used on the hand perforator to the more complicated but shorter code acceptable to the machine, and of comparing two tapes independently hand-perforated by two operators so as to check against errors. They can be used for other purposes, such as making duplicates of tapes, making minor corrections in tapes, making printed copies of the contents of tapes, etc.

b. Reliability.—It has been mentioned before that reliability is the most outstanding operating characteristic of this machine. During the first four months of its operation, the machine of the Ordnance Department gave one wrong answer due to a mistake in the permanent wiring and a few wrong answers due to operating or maintenance personnel interfering with the operation of the machine. Not a single wrong answer was obtained as a result of transient machine trouble, such as relays failing to give a contact.

The reason for this performance is not that transient trouble does not occur—it does, in fact, occur more frequently than on many other machines —but that it is rendered harmless by the elaborate checking system built into the machine.

The checking system has been already mentioned several times, especially in sections 2d and 4b. Those instances are not exhaustive. One or more checks are provided for almost every operation of the machine. As an example, let us consider the process of reading a code from a tape. As mentioned in section 4b, the tape transmitter moves the tape until a code is exactly above the sensing fingers. Then the tape is stopped, the fingers come up, and those fingers which find holes in the tape penetrate them, thereby closing certain electrical contacts and breaking certain others. The electric paths thus closed set up the corresponding "finger relays." There are in all six finger relays, corresponding to the possible six holes in the tape.

A circuit, previously set up, checks whether the right number of finger relays is up, i.e., whether the code just read consists of the expected number of holes. The machine then translates the code; i.e., it operates a binary and quinary relay in case of a number, or a single relay corresponding to the code read in case of an order. It checks the translation by making sure that one and only one relay in a proper group is up. It then sets up the "steering relays" which will direct the translation of the next code into the proper relay group (for example, when reading numbers, the steering relays direct the first code into the bi-quinary relays representing the first digit, the second code into the relays representing the second digit, etc.). The steering relays which were used for the code just read are dropped out. Relays are set up which will determine the proper number of holes expected in the next code. These and, in some cases, other steps are performed approximately in this order.

If any one in this series of steps fails, the machine stops and gives an alarm. The stoppage may be due to equipment failure (the most frequent being dirt between contact points of a relay, which prevents a circuit from being closed), an extra hole in a tape caused by long wear, an operator's error in perforating an improper code in a tape, or the like.

The system of checks and safeguards just described does not apply only to the reading of numbers or orders from tape but is typical of all operations of the machine. For instance, checks are performed at the end of each addition, and at the end of each of the partial additions of which multiplication is composed, to make sure that the proper number of relays is operated. Before any number is transferred into a register, a "down check" is made to insure that all relays of the register are in their released state. Before a number is taken out of a register, an "up check" insures that at least one binary and one quinary relay for each digit are operated. A permanent check is kept against "over-registration" in all registers, so that the machine is stopped instantly when more than two relays come up in the same digit at any time. Down checks are taken before registering an order in the routine control, table control, etc. These and many other checks have the common feature of stopping the machine instantly when a failure occurs. If the machine were allowed to proceed, wrong answers might result; by stopping it, not only are wrong answers prevented from arising but the task of finding the source of the trouble is made enormously easier.

If a trouble is encountered, one of two things happens depending on the setting of a key. One setting, designed for daytime operation in the presence of operating personnel, causes the machine to stop and sound an alarm. The other setting, designed for unattended operation, causes it to act as if a section of the problem had been finished: the control is returned from the routine tape to the problem tape, and the latter is moved to the beginning of the next (minor or major) section. If no more sections are provided on the problem tape, the computer is disconnected from the position and connected to some other loaded position. This arrangement makes it possible to obtain the fullest use of machine time during unattended hours.

c. Locating and Remedying Machine Trouble.—Each computer contains a panel of about 400 lamps which indicate the progress of the problem and

the numbers stored in various places. When a machine stoppage occurs, a quick scanning of this control panel indicates the nature of the trouble and the computational step in which the trouble has occurred. Since each step involves only a relatively small number of relays, it is easy to find the first relay which has failed to come up or has come up through error. It remains to investigate the circuit which energizes that relay. This circuit is likely to be long. Nevertheless, if it is one of the more frequently used circuits, the maintenance man probably knows its course and can check systematically the several dozen contact points through which this circuit flows. If it is an unfamiliar circuit, he may have to consult the circuit diagrams. In the former case, locating and remedying the trouble is accomplished in a few minutes; in the latter case it may take as long as two or three hours. From past experience, the average time required is ten minutes.

As for the frequency of stoppages, the experience gained during the initial period after completion of the machines cannot be considered as typical. During this period, various unfavorable conditions led to failures with a frequency ranging up to ten or twenty per day. An extrapolation of the experience gained from the smaller relay-type machines of the Bell Telephone Laboratories leads to the expectation that the frequency of machine failures will eventually be at most one per day, and possibly as low as one or two per week.

The number of stoppages caused by operators—usually by improperly perforated tapes—is expected to be higher than that of machine failures. Depending on the amount of tape-making called for by various problems, the frequency of such errors is expected to vary somewhere between one and ten per day. These estimates are naturally tentative. Operator errors are usually easier to trace and to remedy than machine failures. Cases in which the machine is stopped for more than five minutes because of such an error should be infrequent.

d. Personnel Requirements. Coding.—A two-computer installation requires for full efficient operation the services of one mathematician in charge, one maintenance man, and approximately four operators. Depending on the types of problems handled and similar circumstances, it might be advisable to have a substitute maintenance man and some clerical help, as well as additional machine operators.

The duties of the machine operators include not only the physical operations such as pressing the start keys, loading tapes into the machine, or hand-perforating of tapes (in fact, these duties occupy only a minor fraction of their time), but mainly the composition of problems, i.e., the translation of the mathematical formulae which govern the problem into the codes which are read by the machine. Depending on their mathematical ability, the operators can assume a large part of the duties of the mathematician in charge, which consist in dealing with such mathematical questions as the selection of appropriate methods of numerical analysis, selection of limits of accuracy, control of the rounding errors occurring in a problem, and the like.

The coding of problems (i.e., the translation mentioned before) is a simple and straightforward process. It is hardly possible to give any estimate of the time required for it, since it is inevitably interwoven with the mathe-

matical preparation of the problem. If the desired course of the computation were exactly known in advance, the operator could code the problem as fast as he (or she) can write—about ten seconds per order, or less than an hour for most problems. But, in practice, the operator is concerned with numerous questions of digital accuracy and the accumulation of rounding errors, with efficient ways of storing intermediate results so as to minimize computing time, with choices between several possible numerical methods to achieve the same result, with scale factors occasionally needed to keep all numbers within the capacity of the machine, and with the proper arrangement of the output of the machine. Including all these elements, which constitute the mathematical preparation of the problem, the time required for preparing a problem has ranged from half a day to a month and may occasionally be even longer in the case of complicated problems. Certain features of the Bell Laboratories' machine, such as the floating decimal point, tend to facilitate the task of the operator or mathematician in comparison with other machines.

The training of operators is surprisingly easy. A person with mathematical experience equivalent to an M.A. degree, or a B.A. degree and several years of experience in computing, requires between one and three months of training to become fully conversant with the machine and competent in its operation, and perhaps another three months to become fast and efficient. The ease of training personnel is an important advantage of the machine.

On the other hand, the training of maintenance personnel for this machine is not easy. The circuits are complicated, and yet it is desirable that the maintenance men know most of them by heart. A period of from six to twelve months seems to be needed for fully adequate training.

e. Speed.—Enough has been said in the preceding pages to arrive at an appreciation of the speed or lack of speed of this machine. Of all the machines which have been developed in recent years, possibly with the exception of the Harvard "Mark I," this is the slowest.

The calculator will perform an addition in 0.3 second, a multiplication in 1 second, a division or square root in perhaps five seconds on the average (the exact computing time depends on the digits occurring in the operation). However, the routine control requires about two seconds to read the order for any of these operations. Thus the calculator remains idle part of the time, and the reading of orders becomes the controlling element in determining the time requirements of a problem.

Printing or perforating a number takes about three seconds. However, while the printer control carries out this operation, the routine can go ahead with the reading and execution of the next order. Thus, unless several printing or perforating orders follow each other (which can usually be avoided with a little care in programming), the computing time is not increased by the slowness of these operations.

"Hunting" for a section of the routine tape cannot be overlapped by reading of routine orders and therefore adds to the total computing time. By careful programming, the amount of hunting can usually be reduced to a point where it constitutes only a small fraction of the total computing time.

The speed of hunting is 2 inches per second, or about $\frac{1}{2}$ second for each order which is skipped in hunting.

Finding a desired location on a table tape is potentially the most time-consuming operation of the machine. It can be overlapped by reading of routine orders, computing, printing, or any other operation not involving the table register and table control. In composing a problem, every effort is made to give tape orders as early as possible, so as to allow ample time for the machine to hunt for the desired location on the table tape. In most of the problems which have so far been done on the machine, it has been possible to overlap tape hunting completely with other operations so that no loss in machine time resulted. There are, however, problems where this is not feasible and where a considerable part of the total machine time is spent in tape hunting. This condition could have been remedied to a large extent if each computer had been equipped with three storage tapes instead of only one.

With the exception of problems requiring unusually long hunts, then, the computing time is about two seconds per routine order. In the problems handled so far, a little more than half of all the routine orders referred to genuine arithmetic operations, and the remainder to tape reading, printing, number transfer, clearing of registers, discrimination, and other operations not usually considered as separate steps in computing. This statement "a little more than half" is an important characteristic of this machine; the proportion may be quite different on other machines. Furthermore, on some machines it may be necessary to insert arithmetic operations for the purpose of checking the results obtained; on this machine, because of its reliability, this is hardly ever done. Some machines, as explained in section 5a, require several orders for each arithmetic operation.

Thus, in general, the computing time needed for solving a problem on our machine can be estimated by allowing two seconds for each routine order, or about $3\frac{1}{2}$ to 4 seconds for each arithmetic operation. If we prefer to count only the number of multiplications in a problem (the number of divisions and square roots is usually negligible), we may make use of the fact that usually a little more than half of all arithmetic operations in a problem are multiplications. We then have to allow between 6 and 8 seconds for each multiplication.

In comparison with hand computing, the speed of this machine is about five times that of a human computer equipped with a good (fully automatic) desk machine. Since the machine can usually handle two problems simultaneously, its output, hour for hour, is equivalent to that of ten human computers. This ratio is further increased if the machine is left running at night. At this rate it will accomplish the work of thirty computers and will require a crew of about five (we do not count the mathematician in charge since his services would be needed to the same extent for supervising the work of the human computers). The comparison may be further shifted in favor of the machine by the fact that, while the machine loses some time because of stoppages, human computers lose a great deal more because of tiring and similar factors. Also, the work of human computers is far more subject to error than that of the machine; and finally, the machine delivers its output in printed form, which obviates the necessity of much copying, typewriting, and proofreading.

7. Problems for the Machine.

a. Types of Problems Handled.—The list of problems given here is not exhaustive. Rather, it contains a selection of problems which have been carried out or programmed and which are typical of what the machine is capable of doing.

(1) *Systems of linear equations.* Several methods of solution have been tried out on the machine. In all of them, elimination is used to arrive at a first approximation of the solution, and if this is not close enough (because of the cumulative effects of rounding errors), iterations are used to find better approximations. Systems up to 13 equations can be handled efficiently on a single computer. The computing time for the elimination process for 13 equations is about $3\frac{1}{2}$ hours. Division of matrices of the same order can be accomplished by a repetition of the elimination process, taking advantage of the fact that many steps in the computation are common to all repetitions and need not be carried out each time. If the order of the system of equations (or of the matrices) is greater than 13, it becomes necessary to treat the problem on the two computers combined. This entails an increase in computing time, so that 13 equations, solved as a two-computer problem, require about 8 hours of computing time. The time required for systems of higher order varies approximately as the cube of the order. This puts a practical limitation on the size of systems to be solved. A more stringent limitation is set by the seven-digit accuracy of the machine. It is believed that this will limit the process used, even if used iteratively, to about 20 or 30 unknowns.

(2) *Algebraic equations with complex coefficients.* A routine has been set up (but not yet tested) for such problems. The method used is that of "false position."

(3) *Systems of ordinary differential equations.* These have so far furnished the main portion of problems for the machine. Both PICARD's method and step-by-step methods have been tried, and the latter have so far been found more efficient. As an example, in a system of order five each step required about three minutes. This will, of course, vary greatly with the complexity of the system, the length of step, etc. The machine can be directed to change the length of step, as has been mentioned above (section 5e).

With regard to complexity of such problems, the machine can handle systems which are beyond the capacity of existing differential analyzers and of the ENIAC. In fact, there is no reason why systems of very high order could not be handled on this machine.

(4) *Partial differential equations.* Two problems of this kind have been treated so far. In both cases, the region in which the solution was sought was a semi-infinite strip in the plane. The first was a relatively simple equation, describing the magnetization of a thin layer under the influence of a known field at the boundaries of the layer. This known field varies in time but is, at any moment, constant over the boundary. The boundary conditions, and therefore the solution, are symmetric with respect to the central plane of the layer, so that the computation had to be carried out for only half of the layer. The numerical integration was carried out, using step-by-step procedure, for a grid containing two points in the direction of x (across half of the layer) and several hundred points in the direction of t (time). The computing time was about a day, or about one minute per grid

point. The problem was repeated with a grid of five points in the x direction and well over a thousand in the t direction, with a correspondingly longer computing time. The second problem consisted of a more complicated system of equations, and the grid used contained sixteen points in one direction and a very large number of points in the other. The computing time on this problem was about two minutes per point. It is believed that partial differential equations in three independent variables are within the capacity of the machine.

(5) *Computation of functions.* This has been done in a variety of cases. Quadratures, infinite series which are broken off after a finite number of terms, and recursive formulae are the usual methods. They have been applied to the computation of convolution integrals, of Fourier summations and power series, and of a table of the binomial distribution

$$B_x = \binom{n}{x} p^x (1 - p)^{n-x}$$

and

$$S_x = \sum_{i=0}^{x} B_i$$

for $x = 0(1)n$; $p = .01(.01).50$; and $n = 50(5)100$.

(6) *The machine is useful for differencing and for interpolation;* in particular it has been used for third order (Lagrangean) interpolation with unequal intervals, a process which is very tedious by hand computing methods. Other fields of useful application include transformation of coordinates in space, such as occur in photographic surveying, and more involved transformations such as the determination of the location of a point when the sums of distances from certain fixed points are known.

b. Efficient Distribution of Problems Among Large Computing Machines.—We shall attempt to state the requirements which a problem must fulfill in order that its solution on this machine may be more efficient than on other large machines. Such statements will necessarily be vague and tentative.

A necessary condition in order that a problem may be handled efficiently on any large machine, in preference to computing by hand and with desk machines, is that the input of orders and numbers be small in comparison with the number of operations performed. Otherwise the time required to put the problem on the machine becomes too long. As far as numbers are concerned, this requirement means that each number introduced into the machine should be used repeatedly, or at least that intermediate results derived from the original inputs should be used to a large extent. Concerning orders, it may mean either that many problems of the same type (with the same input of orders) are to be solved, or that the problem contains a repetitive nucleus, as is the case in step-by-step methods of integration or in iterative or recursive processes.

The allocation of problems between the several large machines in existence is based on the capacities and time requirements of these machines. With respect to capacity, the usual limiting factor is the capacity to store

numbers; occasionally, the capacity to handle complicated instructions or the digital accuracy have to be considered. With respect to time require-ments, we have to distinguish between (a) set-up time off the machine, (b) set-up time on the machine, and (c) computing time.

Our machine excels in storage capacity (especially if the storage tape is used), in the ability to handle complicated instructions, and in the fact that its set-up time "on the machine" is zero. That is to say, all of the prepa-ration necessary for a problem can be done while the machine is doing some-thing else. Its worst feature is the long computing time required. With respect to set-up time off the machine (the time required for coding and producing of the required tapes, punched cards, or the like, as well as handling of any auxiliary equipment separate from the machine, such as removable plugboards on IBM machines and the ENIAC or the tape processor on our machine), none of the existing or projected future machines seems to have a distinct advantage over any other.

The advantage of the Bell Laboratories' machine in requiring no set-up time on the machine is important in problems for which many solutions are desired intermittently; for instance, evaluation of experimental data which should be made as quickly as possible after the data have come in, or research problems which will be run a number of times with small modifica-tions in the input, the changes to be made each time depending on the previous answers. Problems of this type will preferably be done on this machine unless they are so long that the computing time becomes pro-hibitive.

The next-shortest set-up time on the machine is found on the IBM relay multipliers, where setting up consists merely in inserting two pre-viously wired plug boards and setting a few switches, and can be accom-plished in less than ten minutes. These machines are several times faster than the Bell Laboratories' machine and can therefore be used to good ad-vantage in cases where the computing time is an important consideration. However, they are severely limited in the complexity of problems which they can handle, somewhat limited in storage capacity, and greatly handi-capped by lack of reliability. The Harvard "Mark I" and the Dahlgren "Mark II," have set-up times of less than an hour, combined with great flexibility and considerable storage capacity. Their computing speed is some-what between that of the IBM relay multipliers and that of the Bell Labora-tories' machine. The ENIAC, with a speed perhaps a thousand times greater than that of the Bell Laboratories' machine, has the longest set-up time, of the order of one day. It will be used preferably for problems which can stay on the machine for a long time without requiring any change in the set-up, that is, where the same computing routine is used on a varying numerical input many times in succession.

None of the projected future machines is planned with the automatic set-up feature of the Bell Laboratories' machine (nor are any of these ma-chines contemplated for unattended night operation). However, the set-up times will be moderate, amounting perhaps to a few minutes in each case.

The Bell Laboratories' machine, in addition to being efficient for prob-lems which require frequent setting up, is at present our only means for dealing with problems exceeding the storage capacity of other machines. It is also unexcelled in flexibility (i.e., the ability to handle complicated

routines), although the Dahlgren "Mark II" is approximately equal to it in this respect.

To sum up: This Bell Laboratories' machine and the ENIAC represent two extremes, with the other existing machines fitting in between. The ENIAC is adapted to very long problems, provided they are not too complicated and do not need too much storage capacity. The Bell Laboratories' machine will handle the most complicated problems, requiring considerable number storage, provided they are not too long. The ENIAC prefers continuous runs, the Bell Laboratories' machine does not mind "on-and-off" problems. None of the existing machines will handle problems which are long and require a great deal of number storage. Future machines are expected to fill this gap.

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[1] This machine was built and largely designed by the IBM.

RECENT MATHEMATICAL TABLES

476[A, B].—PETER BARLOW, *Tables des Carrés-Cubes, Racines Carrées, Racines Cubiques et Inverses de tous les nombres entiers de 1 jusqu'à 10.000.* Paris et Liége, Béranger, 1946, iv, 200 p. 10.7 × 17 cm. Bound in boards. 120 francs.

This is, apparently, a facsimile reproduction of one of the stereotyped editions of Barlow's tables edited by AUGUSTUS DE MORGAN, with headings of the columns in French instead of the English originals. See *MTAC*, v. 1, p. 16–17, 26, 100, 169, 356–357; v. 2, p. 85.

477[A, B, C].—EMILIO CAZZOLA, *Tavole Grafiche dei Logaritmi a 6 decimali con interpolazione ottica sulla 6ᵃ cifra Mantisse dei Logaritmi e dei cologaritmi Immediate per numeri da 1 a 100.000 Antilogaritmi Immediati per mantisse da 00.000 a 99.999. (Brevetto N. 415.030.) Tavole Aritmetiche e Numeriche ad uso degli Ingegneri, Matematici, Fisici, Chimici, Geometri, Periti, ecc. nonchè delle Scuole Medie Superiori (Licei ed Istituti tecnici).* Milan, Hoepli, 1947, xix, 90 p. 18.5× 26.1 cm. Stiff paper covers, 1000 lire.

The principal table of the volume (p. 1–63) gives log $N, N = 1(1)100\,000$ to 6D. For $N = 1(1)999$ the mantissae are given in the usual manner. But for $N > 999$, numbers of more than 4 digits are found to correspond to a certain mark or point on a vertical scale; opposite this mark on an adjacent mantissa scale the six-place mantissa may then be read off, the first four digits being given and the last two read off from the scale. Adjacent to each printed four-figure mantissa is given the corresponding cologarithm decimal part. Thus the author claims that in his 63-page table, 8 columns to the page, one has a 6D logarithmic and antilogarithmic table, without interpolation, for 5-figure numbers. In the sixth digit, however, there is, at times, decided uncertainty, to the extent, perhaps, of 4 units. PIERO CALDIROLA, professor of physics at the University of Pavia, praises this work, in an introductory note, as one originally conceived by the author. It may be the first six-place table of the kind, but as long ago as 1925, a five-place table of exactly this same type (but without cologarithms) was published in New York, Macmillan, by ADRIEN LACROIX & CHARLES L. RAGOT. Their work was entitled *A Graphic Table combining Logarithms and Anti-logarithms Giving directly without Interpolation the Logarithm to five places of all five-place numbers and the numbers to five places corresponding to all five-place logarithms, also a*

graphic table as above reading to four places. 48 p. 17.2 × 24.6 cm. The scales are arranged horizontally.

In Cazzola's book (p. 64–65) is a table giving 7D ln N, equivalents of log $N = 1(.01)10.09$; and on p. 66–67, 7D log N, equivalents of ln $N = 1(.01)10.09$. On p. 71–85 are tables of n^2, n^3, $n^{\frac{1}{2}}$, $n^{\frac{1}{3}}$, $1000/n$, πn, $\frac{1}{4}\pi n^2$, $n = 1(1)1000$. On p. 85 are also 6D tables of π and e, their multiples, powers, and logarithms. On p. 86–87 are tables of n^p, $n = 1(1)100$, $p = 4(1)9$; also tables of $n!$, log $n!$, colog $n!$, $n = 1(1)20$. On p. 88–89 are the prime factors of the numbers not divisible by 2, 3, 5, 11, for 91 to 9373; and on p. 90 a table of primes from 2 to 11677. Two of five illustrations on p. "xi" are erroneous.

R. C. A.

478[A, B, C, D].—FRIEDRICH GUSTAV GAUSS, (a) *Techniker-Tafel. Allgemeine Zahlentafeln und vierstellige trigonometrische und logarithmische Tafeln. Ausgabe für technische Schulen und Praxis.* Edited by HANS HEINRICH GOBBIN, grandson of the author. Sixth to tenth editions. Stuttgart, Wittwer, 1947, iv, 105 p. 15 × 23.2 cm. (b) *Fünfstellige vollständige logarithmische und trigonometrische Tafeln (sexagesimal unterteilter Altgrad). Zum Gebrauche für Schule und Praxis . . . Neu herausgegeben von Dr.-Ing.* GOBBIN. 316–320. *Auflage*, Stuttgart, Wittwer, 1946, ii, 184, xxxiv p. 15.1 × 23.6 cm.

Many elementary tables prepared by Gauss, some of them in many editions, have been published during the past 77 years. Apart from brief reviews below we note some facts with regard to such editions and some other five-place tables. (a) Here are included tables of (i) n^2, n^3, $n^{\frac{1}{2}}$, $n^{\frac{1}{3}}$, $1000/n$, πn, $\frac{1}{4}\pi n^2$, $n = 0(1)1000$; (ii) natural trigonometric functions sin, tan, cot, cos for each sexagesimal minute, to 6D, 0 to 5° and 85° to 90°, and to 4D for the rest of the quadrant; (iii) log n, $n = [1(1)10009; 4D]$ and $n = [10000(1)11009; 7D]$; (iv) log sin, and log tan, for $[0(10'')2°; 4D]$, log sin, log tan, log cot, log cos, for each $1'$ of the quadrant, to 4D; (v) ln N, $N = [1(1)1000; 4D]$; (vi) multiples of M and M^{-1}; small tables of $e^{\pm x}$. The first to fifth editions were in 1932. (b) The 5D tables include (p. 2–21) log N, $N = 1(1)$-10009, and S and T for $1'(1')3°4'$; there is also a 7D table (p. 22–23) for $N = 10000$–11009; various functions of π (p. 24). Log sin [cos] and log tan [cot] for each $1''$, 0 to 1° [89°–90°] (p. 25–37); for each $10''$, 1° to 8° [82°–89°] (p. 38–50); log sin, log tan, log cot, log cos, for $0(1')45°$ and S and T for $0(1')3°$ (p. 51–96). Addition and subtraction logarithms (p. 97–108). Ln N, $N = 1(1)1109$ (p. 109–111). Multiples of M and M^{-1}, arc sin, arc cos, arc tan, arc cot (p. 113). Natural trigonometric functions, chords and heights and lengths of arcs (p. 114–124). N^2, $N = [.001(.001)10.009; 4D]$ (p. 125–145). T. IX (p. 146–151) Interpolation; T. X–XI, Weights and measures (p. 151–154); T. XII (p. 154–161), The Earth spheroid; T. XIV (p. 164–171) Constants of nature; T. XV (p. 172–181) Astronomy; T. XVI (p. 182–183) Barometric measurements (p. 184). In the 261st edition, p. 151–184 were very thoroughly revised.

The first edition of (b) was in 1870, vii, 38, 142, [1] p.; the second in 1871; the third in 1872; the fourth in 1873; 14, 1881; 17, 1882; 21 and 22, 1884; 25, 1886; 35, 1891; 44, 1894; 47, 1895; 54–57, 1898; 60–67, 1900; 71, 1902; 80–83, 1905; 84–87, 1905; 101–105, 1909; 111–115, Halle, 1911; 116–125, 1912; 176–190, 1920; 260, 1931; 261, 1935; 271–280, 1937; 281–290, 1939; 301–310, 1943. Of (b) there was a *Kleine Ausgabe* in 1873; third ed. 1891; fifth 1893; sixth 1895; seventh 1896; twelfth 1901; 24, 1906; 29–33, 1910; 43, 1913; 59–63, 1920; 90, 1931.

Two other five-place tables of Gauss were the following:

Fünfstellige logarithmisch-trigonometrische Tafeln für Dezimalteilung des Quadranten, first published in Berlin, by Rauh, 1873; second ed. 1898; third ed. 1904; fifth-sixth, 1926; seventh–eighth, 1937, of which there was a new ninth–tenth ed. with *desimal unterteilter Neugrad*, ed. by Gobbin, 1940.

Fünfstellige vollständige trigonometrische und polygonometrische Tafeln für Maschinenrechnen.
Halle, 1901, 100, xviii p.; Stuttgart, second ed. 1912; fourth-fifth 1925; sixth–seventh 1934; eighth–tenth 1938.

Friedrich Gustav Gauss, often called Kataster-Gauss [Surveyor-Gauss] was born 20 June 1829 in Bielefeld, Germany, and died, 26 June 1915, in Berlin, as Generalinspektor des Preussischen Katasters (E. HARBERT, *Vermessungskunde*, v. 1, third ed., Berlin, 1943, p. 276). He has been confused with the Gymnasium mathematics teacher ALEXANDER FRIEDRICH GUSTAV THEODOR GAUSS (b. 1831, see POGGENDORFF, v. 4) to whom (1) all of the books of F. G. Gauss are credited (Dec. 1947) in the Catalogue of the Library of Columbia University; (2) F. G. Gauss's *Polygonometrische Tafeln*, Halle, 1893, is credited on p. 385 of E. WÖLFFING, *Mathematischer Bücherschatz*, Leipzig, 1903.

In the Library Catalogue of Brown University (Nov. 1947) he has also been confused with C. F. GAUSS, in cataloguing the Russian tables, B. NUMEROV, *Tables for Calculation of Geographic and Rectangular Coordinates of Gauss-Krüger*, Leningrad, 1933. C. F. Gauss's work prior to 1820 was in this connection developed by J. H. L. KRÜGER (1857–1923); see E. HARBERT, *ibid.*, p. 92. In our review of tables by PETERS, *MTAC*, v. 2, p. 299, we referred to his table for the improvement of the "Gauss-Krüger" projection. See also elsewhere in this issue, OAC, Bibliography Z–III 13.

R. C. A.

479[A, B, C, D, E].—L. J. COMRIE, *Chambers's Four-Figure Mathematical Tables.* Edinburgh & London, W. & R. Chambers Ltd., 1947, iv, 64 p. 19.2 × 25.5 cm. Limp linen cover. 5 shillings.

"These tables represent a deliberate attempt to raise the standard of the four-figure mathematical tables used in the highest school classes, and in technical colleges and universities, as well as in industrial practice. Most of the existing compilations have shown a tendency to follow conventional paths and thus lag behind the progress made by applied mathematicians, physicists, and engineers." In this way Dr. Comrie commences his Preface. Innovations of his presentation are so numerous in the admirably clear, closely packed pages, with excellent typographical displays throughout, that it is not easy in brief space to give an adequate idea of the contents. They are evidently arranged by a master of computational methods in practice. The presentation is naturally somewhat reminiscent of the contents of *Standard Four-Figure Mathematical Tables* by L. M. MILNE-THOMSON & L. J. COMRIE, London, 1931.

In connection with tables of (a) trigonometric and (b) circular functions only two types of tables are used, one being for (a) at interval 0°.1 or 6′, with proportional parts for 0°.01 and 1′, and the other for (b), with radian argument. In the tabulation of the trigonometric functions the method of complementary arguments has been used so that no separate tabulation is given of co-functions.

We do not recall any other table where quite the same care and clarity are displayed in guiding the reader in the use of tables of S, T, $\sigma = x \csc x$, $\tau = x \cot x$, $Sh = \log (\sinh x/x)$, $Th = \log (\tanh x/x)$, σh, τh, critical table(s) (C. T.). On consecutive pairs of pages (10–35) are the following:

Log sin, 0(1′)17°; C.T. 81°–90°; S for minutes. Log sin, 0(0°.01)11°; C.T. 81°–90°; S for degrees. Log sin, 10°(0°.1)90°. Log tan, 0(1′)18°; T for minutes. Log tan, 0(0°.01)11°; T for degrees. Log tan, 10°(0°.1)90°. Nat. sin, 0(0°.1)90°; C.T. near 0 and 90°. Nat. tan, 0(0°.1)80°.

Nat. tan, 79°(0°.01)90°; τ for degrees. Nat. tan, 72°(1′)90°; τ for minutes. Nat. sec, 0(0°.1)80°. Nat. sec 79°(0°.01)90°; C.T. to 5°; σ for degrees. Nat. sec, 72°(1′)90°; C.T. to 5°; σ for minutes. It is pointed out that the advantages of critical tables are (1) they do not need interpolation, and (2) they always give results that are correct to within half a unit of the last decimal.

Then on p. 50–62, are tables of circular functions $0(0^r.01)1^r.57$ with auxiliary tables; τ and σ for circular cotangents and cosecants; circular and hyperbolic functions $0(.001).2$; exponential and hyperbolic functions $0(.01)3(.1)5.4$; τh, σh, etc.

All that is necessary for exact interpolation is carefully provided. On pages ii, iii of the cover are proportional parts for differences $1(1)201$. Differences greater than 201 may be dealt with by compounding, or by using a slide rule.

Inverse circular and hyperbolic functions are not explicitly tabulated, but it is suggested that they may be obtained by inverse interpolation, except in the case of large functions, when the same method would be applied to the reciprocals of the functions. Interpolation with second differences is introduced with an example on p. 49.

Other tables are of squares, square roots, reciprocals, natural logarithms, prime numbers, factors, binomial coefficients, and probability integral.

The volume has also many footnotes connected with the tables given on the same page. For example: Derivatives and integrals involving powers (p. 39); Common and natural logarithms (p. 41); Evaluation of $x = a^b$ (p. 43–45); Series for (i) circular functions, (ii) inverse circular functions, (iii) hyperbolic functions, also Series, derivatives and integrals for exponential functions, Derivatives, integrals and log series for hyperbolic functions, and Formulae and series for inverse hyperbolic functions (p. 50–60).

R. C. A.

480[A, R].—J. J. Levallois, "Compensation des réseaux géodesiques par la méthode des gisements," L'Assoc. Intern. de Géodésie, *Bull. Géodésique*, 1947, no. 3, Jan., p. 49–82; tables p. 77–80. 16.6 × 24.9 cm.

Table I is of $2(K + x - 1)/(K + x)$, for $K = 1(1)20$, $x = [.92(.002)1 ; 4D]$. Table II is for $6(K + x - 1)/[4(K + x - 1) + 3]$ and of $[8(K + x - 1) + 6]/[8(K + x - 1) + 7]$, for $K = 3(1)20$, and $x = [.92(.01)1 ; 4D]$.

481[B].—A. Albert Gloden & J. Bonneau, *Table des Bicarrés des entiers de 5 001–10 000*. Luxembourg, 30 October 1947, i, 21 leaves. 28.5 × 28.5 cm. Typescript one side of each leaf. Edition of three copies made from mss. Other copies will be made, at 30 Belgian francs each, for those applying to A. Gloden, rue Jean Jaurès 11, Luxembourg. There is a copy in the Brown University Library. **B.** A. Gloden, *Table des Bicarrés N^4 pour $3\,001 < N \leqslant 5\,000$*. Luxembourg, the author, 30 October 1947. 19 leaves. 20.2 × 29.5 cm. Mimeographed one side of each leaf. For Gloden's table of N^4, $1000 < N \leqslant 3000$, see *MTAC*, v. 2, p. 250. Thus there are now tables of N^4, $N = 1(1)10^4$.

A. Here is given a table of N^4, for $5000 < N \leqslant 10\,000$. On each leaf after the first are nine columns: N, N^4, $N + 1250$, $(N + 1250)^4$, $N + 2500$, $(N + 2500)^4$, $N + 3750$, $(N + 3750)^4$, the ninth column containing the last four digits for each of the fourth powers in the line. The table was prepared for verifying factorizations of numbers of the form $N^4 + 1$. Each result was calculated twice, directly, and by the aid of finite differences.

Extracts from text

482[C, D].—Alfred George Cracknell, *Clive's Mathematical Tables containing two-page Tables of Logarithms, Antilogarithms, Natural and Logarithmic Trigonometrical Functions, and Circular Measure*. London, University Tutorial Press, eleventh impression, 1946, ii, 50 p. 14 × 21.4 cm.

The nine tables (p. 31–49) include 5D tables of the six natural and logarithmic trigonometric functions. The first edition of these tables appeared in 1906 and the second impression in 1909.

483[D].—JOHANN ROHRER, *Tachymetrische Hilfstafel für zentesimale Kreisteilung.* Berlin-Grunewald, Wichmann, 1942, 12 p., stiff paper. 16.9 × 24.3 cm.

Pages 3–12 give 4D tables of the values of $\cos^2 \alpha$ and of $\sin \alpha \cos \alpha$, for every 2 centesimal minutes of the quadrant. A more substantial table of the first of these functions was given by E. A. SLOSSE (see *MTAC*, v. 1, p. 38; compare RMT **453**). More extensive tables of each of the functions are given by MARCHISIO (RMT **487**).

484[D, E].—NBSCL, *Table of Circular and Hyperbolic Tangents and Co-tangents for Radian Arguments.* "Second printing," New York, Columbia University Press, [1947], xxxviii, 410 p. 19.6 × 26.5 cm. $7.50. See *MTAC*, v. 1, p. 178–179 (review by D.H.L.).

The second printing (edition of 700 copies) within four years not only of this volume but also of *Table of the Bessel Functions* $J_0(z)$ *and* $J_1(z)$ *for Complex Arguments* (edition of 845 copies), reviewed in our last issue, is a significant indication of the great need which has been met by these admirable publications. The changes from the first edition of the present volume, are as follows: The volume is slighter because the paper is thinner, and doubtless not so durable; the title-page date, 1943, has been removed; on p. ii has been added, "First printing 1943, Second printing 1947"; on p. vi, previously blank, appears "Acknowledgment," a paragraph formerly in the Preface, no longer on p. xi; on p. xi now is a Note stating that the present volume was prepared from the negatives used for the earlier printing except that for p. 316, where the error we noted *MTAC*, v. 2, p. 279, is now corrected; for the two-page list of publications of the NBSCL (following p. 410) are now substituted four new pages, although the incorrect reference to the National Bureau of Standards, rather than to the Superintendent of Documents, for the distribution of items, (1)–(12) in the list, remains. The same criticism may be made of the three footnote items p. xxi. The price of the second printing is fifty percent higher than that of the first.

As in *MTAC*, v. 3, p. 25, we find lack of careful revision of the Bibliography with its inaccuracies, omissions, infelicities, and lack of reference to readily available American editions.

For the work of ADAMS & HIPPISLEY the date of the corrected reprint of 1939 would have been preferable to that of the original edition, 1922—Besides the *siebenstellige* volume of BRANDENBURG there should have been an elimination of the last-line comment, and a reference to the *sechsstellige* work, and also to errors listed in *MTAC*, v. 1, p. 162, 388; v. 2, p. 46–47, 277—The date 1912 instead of 1870 for the English edition of the work of BRUHNS is decidedly misleading (see *MTAC*, v. 2, p. 338–339)—So also for the *Tables Portatives* of CALLET with the remark "There are similar tables by W. GARDINER." Some understanding of the relation of Callet's 1783 work to Gardiner and of what was in Callet's 1795 volume (1860 is the apparently intended as a definitive date in the Bibliography) may be gleaned from HENDERSON—The German edition of *Tafeln elementarer Funktionen* by EMDE is listed, but not the American edition, 1945—To list mils tables of the Fort Sill, Field Artillery School, and to omit reference to the fine volume (1943) prepared under the direction of M. M. FLOOD (see *MTAC*, v. 1, p. 146) seems inexcusable—Since L. J. COMRIE is the well known author of the 1939 *Seven-Figure Trigonometrical Tables, for Every Second of Time* (see *MTAC*, v. 1, p. 43), why was his name suppressed?—On p. xxix, l. 16, *for* 62, *read* 62–64.— The third edition of the *Recueil* of HOÜEL, 1885, is referred to as if it were the first edition— The 1918 edition *Siebenstellige Werte der trigonometrischen Funktionen von Tausendstel zu Tausendstel des Grades* by PETERS, always almost impossible to procure, is listed, but not the public editions of 1930 and 1938, nor the American edition of 1942 (see *MTAC*, v. 1, p. 12–13)—We are told that the second edition of *Sechsstellige Tafel der trigonometrischen Funktionen* was published in 1929 (really the date of the first edition); the second edition did not appear until 1939—The inaccessible 1939 edition of *Achtstellige Tafel der trigonometrischen Funktionen für jede Sexagesimalsekunde des Quadranten* is listed, but there's no

reference to the American edition of 1943 in English—The third edition of *Trigonometriae* by PITISCUS, 1612, is listed as if it were the first edition—The place of publication is lacking in connection with the 1624 (not 1625) table of URSINUS—There are no references to the following tables: M. TAYLOR, *Tables of Logarithms of all Numbers, from 1 to 101 000, and of the Sines and Tangents to every Second of the Quadrant*, London, 1792 (see *MTAC*, v. 1, p. 42); G. STEINBRENNER, *Fünfstellige Trigonometrische Tafeln* . . ., Brunswick, 1914 (see *MTAC*, v. 1, p. 38); and E. A. SLOSSE, *Tables des Valeurs Naturelles des Expressions Trigonométriques* . . ., Tientsin, China, 1923 (*MTAC*, v. 1, p. 38).

R. C. A.

485[D, E, L].—GEORGE WELLINGTON SPENCELEY & RHEBA MURRAY SPENCELEY, *Smithsonian Elliptic Functions Tables* (*Smithsonian Miscellaneous Collections*, v. 109), Publication no. 3863. Washington, Smithsonian Institution, 1947, iv, 366 p. 16 × 23½ cm. See *MTAC*, v. 1, p. 325, 331. $4.50

In these days of large computing installations, one hesitates to describe as monumental even so splendid an achievement as that under review. These tables may be more cautiously described as the most important elliptic tables published up to now. Their nearest rivals are LEGENDRE'S classic tables of elliptic integrals (1816, 1826) and the well-known GREENHILL-HIPPISLEY tables published by the Smithsonian Institution in 1922.

Legendre's tables gave values to 9–10D of the elliptic integrals

$$F(\theta, \phi) = \int_0^\phi (1 - \sin^2 \theta \sin^2 \phi)^{-\frac{1}{2}} d\phi, \qquad E(\theta, \phi) = \int_0^\phi (1 - \sin^2 \theta \sin^2 \phi)^{\frac{1}{2}} d\phi$$

for $\theta = 0(1°)90°$, $\phi = 0(1°)90°$; there were also various single-entry tables, for instance, tables of the complete integrals $F(\theta, \frac{1}{2}\pi)$ and $E(\theta, \frac{1}{2}\pi)$, now usually called K and E respectively. In relation to their time, Legendre's tables were a tremendous achievement, involving a colossal amount of computation of new functions. Moreover, three photographic reprints, by POTIN, EMDE and PEARSON (see *MTAC*, v. 2, p. 136, 137, 181), within the last quarter of a century are testimony to their enduring value. It may be added in passing that elliptic *integrals*, in the Legendre form, are perhaps the most frequently needed elliptic quantities in practical work aiming at numerical results. Legendre's work is, thus, far from superseded; a considerable transfer of popularity to the use of $k^2 = \sin^2 \theta$ instead of θ as modular argument has affected its position, but few would care to dispense with it; it should also be noted that the only tables of *incomplete* integrals with k^2 as the modular argument (namely those of SAMOĬLOVA-ĬAKHONTOVA, 1935) give only 5D and were computed merely by interpolation in Legendre. It cannot, however, be contended that Legendre gives more than a fraction of the information given in the volume under review; all of Legendre's values could be obtained by interpolation in Spenceley, while the reverse is far from true.

With the Greenhill-Hippisley tables the present volume is more closely connected; it may be described as a bigger and better version, tabulating more functions for more arguments to more decimals, but inspired by and closely patterned on the former, as the authors acknowledge in their opening words.

The first to engage in extensive tabulation of theta functions appears to have been J. W. L. GLAISHER[1], under whose superintendence in 1872–75 tables were computed for the Tables Committee of the British Association for the Advancement of Science. The tables were completely set up in type (360 pages), but were not published, and no printed copy appears to have survived; the manuscript, however, still exists. Even had these tables been published, they would have been less important than those of the Spenceleys. They gave no Jacobian elliptic functions or integrals, but simply theta functions, differing from Jacobi's by simple factors. These factors made Glaisher's four theta functions constitute three numerators and a common denominator of the Jacobian elliptic functions sn u, cn u, dn u, so that they were excellent intermediate quantities for the computation of the latter, but as theta functions they were much less convenient than Jacobi's. The Glaisher tables[1] did,

however, have arguments $\theta = 0(1°)89°$, $x = \pi u/2K = 0(1°)90°$, and it is only now, with the publication of the present tables with the same arguments, that one can safely assert that Glaisher's tables have little further value.

In 1911 Greenhill brought forward a scheme for the rearrangement of the British Association elliptic tables. In terms of ϑ's as in the standard text of Whittaker & Watson, his theta functions may be defined by

$$A(r) = \vartheta_1(x)/\vartheta_2, \qquad B(r) = \vartheta_2(x)/\vartheta_2, \qquad C(r) = \vartheta_3(x)/\vartheta_4, \qquad D(r) = \vartheta_4(x)/\vartheta_4,$$

where $x = \pi u/2K = r°$. These have the advantage, for what it may be worth, that while simple relationships of the kind $B(r) = A(90 - r)$, $C(r) = D(90 - r)$ are retained, the division-values arising at bisection, trisection, etc. of the quadrant (for x) or of the quarter-period K (for u) are algebraic functions of the modulus. For each modulus and for $r = 0(1)90$, Greenhill proposed to tabulate the four theta functions $A(r), \cdots, D(r)$, together with the Jacobian amplitude ϕ defined by $u = F(\theta, \phi)$ and the Jacobian zeta function (or periodic part of the second integral $\int_0^u \mathrm{dn}^2 u\, du$), which unfortunately he called $E(r)$. After some specimen tables had been published in the British Association *Reports* for 1911, 1912, 1913 and 1919, tables of this kind calculated by Hippisley were published by the Smithsonian Institution in 1922, with an introduction by Greenhill. These tables give one opening (two pages) to each of the modular angles $\theta = 5°(5°)80°(1°)89°$; the column headings on left pages are r, $F\phi$, ϕ, $E(r)$, $D(r)$, $A(r)$; the arrangement is semi-quadrantal, and other functions appearing in column headings on right pages are merely the complementary forms of the above, for example, $B(r)$ means $A(90 - r)$. The argument r takes values $0(1)90$; $F\phi$ means simply u, $= rK/90$, and is a subsidiary argument. The values are to 10D, except that ϕ, regrettably, is given only to the nearest minute. In the headings for each opening are given the values of K, K', E, E', q, $\Theta 0[= \vartheta_4]$, $HK[= \vartheta_2]$ for the corresponding θ.

Little space is occupied in the Spenceley volume by a laconic preface and an equally laconic appendix; one would be interested to know what machines, presumably desk calculators, were used. The magnificent main table occupies p. 2–357. There are four pages to each of the modular angles $\theta = 1°(1°)89°$. The tabulation is quadrantal. The column headings are

r, $u = (r/90)K = F(\phi, k)$, sn u, cn u, dn u on left pages,
r, ϕ, $E(\phi, k)$, $A(r)$, $D(r)$ on right pages.

On alternate openings, $r = 0(1)45$ and $45(1)90$. All function values are to 12D. u is a subsidiary argument, as in Hippisley. The Jacobian elliptic functions sn u, cn u, dn u, not tabulated by Hippisley, nor to anything approaching the present extent by anyone else, were apparently computed as quotients of theta functions. ϕ is given to 12D in *radian* measure. The number of decimals in ϕ fills a great need, though anyone interested in checking Legendre by interpolating in Spenceley will view the radian measure with mixed feelings. $E(\phi, k)$ with the Spenceleys means what it ought to mean, namely the second integral

$$\int_0^\phi (1 - k^2 \sin^2 \phi)^{\frac{1}{2}} d\phi = \int_0^u \mathrm{dn}^2(u, k) du,$$ not the zeta function. $A(r)$ and $D(r)$ are exactly as in Hippisley. The values were calculated to 15D, checked within a few units of the last decimal, and cut down to 12D for publication. Brown University Library has a photographic reproduction of the original 15-place manuscript. In the page headings for every argument are given the values of K, K', E, E', $D(90)[= 1/\sqrt{k'}]$, $1/D(90)$ to 15D, and of q, q' to 16S.

The appendix gives a number of formulae for computation, and also two tables: p. 364–365, values to 15D of $r\pi/180$, sinh $(r\pi/180)$, cosh $(r\pi/180)$ for $r = 1(1)90$; p. 366, values to 25D of sin $(r\pi/180)$, cos $(r\pi/180)$ for $r = 1(1)89$.

The reviewer has made no numerical checks beyond comparing about a third of the values of K and E with values to about 13D computed by himself. With the checks described it is difficult to see how there can be much error, even though differencing is not mentioned. The latter would have required a considerable effort, as the tables contain about 720 values for each modular angle, or altogether about as many as in a natural trigonometrical table

giving sines and tangents at interval 10″, while the higher differences are much larger in the present tables.

An old-fashioned reviewer, addicted to desk and even to armchair methods, and not disposed to have his standards of measurement completely shot from under him by "what the new machines could do," must consider this a magnificent piece of computing. Professor and Mrs. Spenceley, and the six named National Youth Administration students who shared the work with them in the earlier years, must be heartily congratulated on the final achievement. If a few criticisms and queries follow, one is glad to be able to point out that they relate far more to the Greenhill program than to its splendid implementation.

Let us begin with a point which some may regard as trivial. Greenhill himself made a good start in 1911; his theta functions were to be $A(r°)$, $\cdots$, r a number (integral in the tables). In fact, they could be called $A(x)$, $\cdots$, and x could be measured in other units than degrees. But his notation quickly degenerated into $A(r)$, $\cdots$, and it is evident that Greenhill never made up his mind whether his r was a number or a number of degrees, which for distinction we may call an angle; he used r in both senses cheek by jowl. Matters are not greatly improved by the Spenceleys. It is true that, except in the headings of the small subsidiary tables at the end, their r is incontestably a number; but in the main table their theta functions are $A(r)$, $\cdots$, while in the text of the appendix they are $A(r°)$, $\cdots$. Moreover, their θ is an angle in the main table, both an angle and a number in the appendix. Perhaps this should not be overstressed; positional astronomers, for instance, have often used the convention that x'' can mean not only $x \times 1''$, but also $x/1''$, that is, the number of seconds in an angle x, with only aesthetic loss to their science. Yet I think most mathematicians will agree that ° is best regarded as a useful abbreviation for $\pi/180$, so that, for instance, $(1°)^2 = 1''.097$ approximately; and if the columns of a table are not headed r, sinh $(r\pi/180)$, they will prefer r, sinh $r°$ to $r°$, sinh r.

The appendix contains another small infelicity. The reviewer is quite willing to identify sn $\frac{1}{2}K$, say, with sn $(45K/90)$; but to describe sn $\frac{1}{2}K$ as sn $45°$ is surely an excess of computer's license. Admittedly it is not easy, especially when divisions into 90 parts are used, to find a concise yet respectable notation which will indicate the theta-function argument x in the case of those functions which in analysis customarily have the elliptic-function argument u. There is much less awkwardness when the centesimal system is used (as in small tables by Hoüel[3]), for $x = 50^g$ or $0^g.50$ corresponds to $u = 0.50 K$.

To come to the most important point: was Greenhill wise to tabulate his theta functions in place of the Jacobian ones? Is there any substantial "advantage" in the fact that the division-values are algebraic functions of the modulus? It is plain that Greenhill revelled in the sort of algebra by which division-values may be obtained, and it is easy to see the attractiveness of an independent check, though the Spenceleys appear to have got on quite well without it. But after all, values for only a few arguments are likely to be checked by this process, and in these cases the values could very easily be multiplied by the necessary factors, ϑ_2 or ϑ_4, if Jacobian functions were being tabulated. The whole subject of elliptic functions has long been notorious for its conflicts of notation; we are now faced by a fresh conflict of notation, between theory and tables. It is safe to say that most mathematicians now expound the theory of theta functions in some kind of ϑ notation; are they to convert the tabulated Greenhill values to Jacobian ones, or to rewrite the theory in terms of Greenhill functions? Greenhill went some way in the latter direction; certainly theory and tables cannot remain for ever divorced, but there appears to the reviewer to be a considerable probability that mathematicians will choose the former course. It has already been mentioned that in the Hippisley tables the necessary factors ϑ_2 and ϑ_4 are given in the headings for each modulus. This is the only respect in which the new tables seem to the reviewer to fall short of Hippisley's. Failing the tabulation of Jacobian theta functions, one could wish that ϑ_2 and ϑ_4 had replaced $D(90)$ and its reciprocal in the headings. After all, $D(90)$ is given in the line $r = 90$ (certainly to "only" 12D, but $D(90) = (\cos \theta)^{-\frac{1}{2}}$, and $\cos \theta$ is given to 25D on the last page). As things stand, ϑ_2 and ϑ_4 are presumably to be computed either from $\sqrt{(2kK/\pi)}$ and $\sqrt{(2k'K/\pi)}$ respectively, or from q-series. The loss of convenience is plain.

A small suggestion may be made, in case these valuable tables are reprinted at any time.

It would be a convenience if the information in the headings, besides being distributed throughout the volume, were also brought together on two pages at one opening, say in the appendix. As far as the reviewer is aware, there are no other tables whatever giving values of K, K', E, E' to 15D, or of q, q' to 16S, at interval 1°.

One is glad to see in the preface that a set of 5-place tables is suggested; the reviewer agrees that they would be very useful. He also sympathizes with the convenient notation snh $u = -i$ sn iu, etc. used in the appendix.

Whatever may be the course of the inevitable rapprochement between the theoretical and computational aspects of elliptic functions, it is plain that the scientific world owes a very great debt to the authors and publishers of the volume under review. There will be few readers of $MTAC$ who will not wish to obtain a copy immediately.

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¹ EDITORIAL NOTES: Why the Glaisher Tables of Elliptic Functions were set up in type but remained unpublished, has never been revealed, although certain facts are on record, from Glaisher's pen, in BAAS *Report*, 1873, p. 171–172, 1930, p. 250–251 (also with later information); *Mess. Math.*, v. 6, 1876, p. 112; and H. J. S. SMITH, *Coll. Math. Papers*, v. 1, 1894, p. lxiii–lxiv. Among various letters on mathematical tables (now the property of R. C. A.) which formerly belonged to Glaisher, are 17 (Oct. 2, 1873–Sept. 15, 1881) dealing with these tables. One of these, from Glaisher to the printers, (a) Taylor & Francis, (b) Spottiswoode & Co., was as follows:

1 Dartmouth Place, Blackheath, S. E.
1874 September 16

"Dear Sirs:

"I am directed by the Committee of the British Association on mathematical tables to ask you if you would be so good as to state the cost for which you would be willing to print and stereotype for them a large table of which I enclose one page as a specimen.

"The table consists of 356 pp. similar to that which I enclose but the last two figures in all the columns are throughout to be omitted (so that *e.g.* the 1st, 3rd, 5th, & 7th columns would take only 8 figs. after the decimal point).

"Please notice that certain of the first figures on 178 of the pp. (always 1's and 2's) have a rule over them, which might perhaps require special type to be cast. We should like the size of the page to be an octavo of about the same size as that of the Nautical Almanac, the figures in the Diff. columns to be in smaller type than those in the other columns, and a white line to be left between every five lines.

"Of course we should wish to have as many revises as would be necessary to ensure freedom from error, but as the manuscript is a good one & has been read with the originals I do not anticipate that more than one revise will often be required. A proof of the page after it is stereotyped would also be desired.

"The table would be preceded by an introduction of perhaps 100 pp. of mathematics of a character not very different from Booth's *New Geometrical Methods* (but without figures). Could you also let us know (very roughly) about how much per sheet this may be expected to cost. Of course there are details which I have not mentioned, & which could not be explained without you saw the whole ms. of the table, but the specimen page gives a sufficiently good idea of all the others for all purposes short of an actual contract. The number of copies would be 250."

"Believe me, to be Yours Truly
J. W. L. Glaisher"

The last letter is from Taylor & Francis as follows: "I beg to enclose you samples of paper for the Tables. Will you in returning the one you like kindly let me know how many copies you will have printed."

In connection with this project £809 were spent, of which apparently £259 was for computing and £550 for printing. There was a reference to these tables in *Nature*, v. 15, 1877, p. 252. After Glaisher's death in 1928 30 manuscript volumes covering the work of these ten-place tables became the property of BAASMTC.

What was first planned as a 100-page Introduction to the Tables later developed into a planned "Memoir on the theta and omega functions" by H. J. S. SMITH (1826–1883), to follow the Tables; but Smith died before this was quite complete. It is printed in Smith's *Coll. Math. Papers*, v. 2, 1894, p. 415–621. Four of the above-mentioned 17 letters are from H. J. S. Smith (1873–76) and deal in part with the "Introduction."

² J. Hoüel, *Recueil de Formules et de Tables Numériques*, third ed., Paris, 1885, p. [57]–[59].

486[D, E, P].—I. V. ANAN'EV, *Spravochnik po Raschetu Sobstvennykh Kolebaniĭ Uprugikh Sistem* [Reference book for computations of inherent vibrations of elastic systems]. Moscow and Leningrad, OGIZ, 1946, 223 p. 12.8 × 19.4 cm. Tables p. 173–220. Bound, 9.50 roubles.

T.I: $\tan \alpha$, $\cot \alpha$, $\alpha = [0(.02)6.3(.1)10; 5D]$.

T.II: $\cos \alpha$, $\sin \alpha$, $\cosh \alpha$, $\sinh \alpha$, $S(\alpha) = \frac{1}{2}(\cosh \alpha + \cos \alpha)$,
$T(\alpha) = \frac{1}{2}(\sinh \alpha + \sin \alpha)$, $U(\alpha) = \frac{1}{2}(\cosh \alpha - \cos \alpha)$,
$V(\alpha) = \frac{1}{2}(\sinh \alpha - \sin \alpha)$, for $\alpha = [0(.01)4.99; 5D]$.

T.III: $A(\alpha) = \cosh \alpha \cdot \sin \alpha + \sinh \alpha \cdot \cos \alpha$,
$B(\alpha) = \cosh \alpha \sin \alpha - \sinh \alpha \cos \alpha$, $C(\alpha) = 2 \cosh \alpha \cdot \cos \alpha$,
$S_1(\alpha) = 2 \sinh \alpha \cdot \sin \alpha$, $D(\alpha) = \cosh \alpha \cdot \cos \alpha - 1$, $E(\alpha) = \cosh \alpha \cdot \cos \alpha + 1$,
for $\alpha = [0(.02)10; 5D]$.

T.IV: $F(\alpha) = \alpha (\sin \alpha \cosh \alpha - \cos \alpha \sinh \alpha)/(1 - \cos \alpha \cosh \alpha)$,
$H(\alpha) = \alpha (\sinh \alpha - \sin \alpha)/(1 - \cos \alpha \cosh \alpha)$, $L(\alpha) = \alpha^2 \sin \alpha \cdot \sinh \alpha/(1 - \cos \alpha \cosh \alpha)$,
$N(\alpha) = \alpha^2 (\cosh \alpha - \cos \alpha)/(1 - \cos \alpha \cosh \alpha)$,
$R(\alpha) = \alpha^3 (\sin \alpha \cosh \alpha + \cos \alpha \sinh \alpha)/(1 - \cos \alpha \cosh \alpha)$,
$\Pi(\alpha) = \alpha^3 (\sinh \alpha + \sin \alpha)/(1 - \cos \alpha \sinh \alpha)$, for $\alpha = [0, .5(.01)1(.02)5; 3–5D]$.

Extracts from text

EDITORIAL NOTE: On turning to Professor W. PRAGER, "Tables of certain functions occurring in dynamics of structures," *MTAC*, v. 1, p. 101–103, it will be noted that tables of $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $S_1(\alpha)$, $D(\alpha)$, $E(\alpha)$, have been already discussed. There are 5D tables of $\cosh \alpha \cos \alpha$, $\cosh \alpha \sin \alpha$, $\sinh \alpha \cos \alpha$, $\sinh \alpha \sin \alpha$, in M. BOLL, *Tables Numériques Universelles*, Paris, 1947, p. 476–483, possibly for use in the field of the volume under review.

In the book under review the paper is cheap, the ink is pale, the type of figures is not clear-cut and distinct, so that some figures are not seen at all, and sometimes it is difficult if not impossible to distinguish between a 6 or 9 and 0, or between 3 and 8.

On checking the last four columns of pages 178–180 of Anan'ev's work the following errata were found by S. A. J.:

α	Function	For	Read	α	Function	For	Read
0.46	$V(\alpha)$	0.01625	0.01623	1.21	$S(\alpha)$	1.08934	1.08943
0.53	$T(\alpha)$	0.53024	0.53034	1.23	$U(\alpha)$	0.76196	0.76126
0.60	$T(\alpha)$	0.60074	0.60064	1.25	$U(\alpha)$	0.78658	0.78655
0.67	$S(\alpha)$	1.00830	1.00839	1.27	$T(\alpha)$	1.29750	1.29755
0.79	$V(\alpha)$	0.08228	0.08221	1.30	$U(\alpha)$	0.85123	0.85170
0.85	$T(\alpha)$	0.85380	0.85370	1.36	$U(\alpha)$	0.93336	0.93360
0.97	$V(\alpha)$	0.15297	0.15227	1.40	$V(\alpha)$	0.45933	0.45943
1.03	$T(\alpha)$	1.03953	1.03966	1.41	$T(\alpha)$	1.45655	1.45650
1.11	$S(\alpha)$	1.06333	1.06331				

487[D, P].—PIETRO MARCHISIO, *Tavole pel Tracciamento delle Curve. Tavole Trigonometriche, Tavole Tacheometriche centesimali. Tabelle pel Tracciamento delle Curve con Coordinate Polari e con Coordinate Ortogonali. Introduzione in Lingue Italiana e Francese.* Borgo S. Dalmazzo (Cuneo), Istituto Grafico Bertello, [1947], xxxii, 255 p. 15 × 23.6 cm.

This work is the second edition of the author's *Tavole Trigonometriche Centesimali. Tables Trigonométriques Centésimales. Applicazione pel Tracciamento delle Curve. Tavole pel Tracciamento delle Curve con Coordinate Polari e con Coordinate Ortogonali. Introdusione in lingue Italiana e Francese*, Milan, Libreria Internazionale Ulrico Hoepli, 1930. xxxii, 255 p. + 2 p. "Errata corrige." The 1947 work is a reprint, with type reset, of a corrected 1930 edition, except for the addition of a brief preface to the second edition, and an extension of the main table.

The work was intended primarily for construction of railways and roads, calling for the exact laying out of circular arcs by the use of centesimal tables. The principal tables of the volume, p. 1–201, are 6D tables, for every centesimal minute, of the six trigonometric

functions, versed sin α, versed cos α (= 1 − sin α), arc α, and $\frac{1}{2}\pi - \alpha$; in the present edition are added two more columns, 5D tables of cos² α and sin α cos α. For each grade of the quadrant the values of these 12 functions are given on four pages.

On p. 219–224 is a table for reading off the polar coordinates of ends of arcs (cords of arcs and centesimal angles) of various lengths of arcs from .1 to 100, on circles of varying radii, r = 15(5)30(10)60, 75, 80, 100(25)200(50)500, 600, 750, 800, 1000. There are similar tables for rectangular coordinates, p. 226–227.

Then on p. 229–232, for circles of radius r = [100(50)300(100)2000(500)4000; 9D] are given (a) circumference; (b) arc subtended by 1ᵍ at center; (c) number of grades at center subtended by unit length of the circumference; (d) arcs corresponding to angles at center of 1°, 1′, 1″; (e) number of sexagesimal seconds at center corresponding to unit lengths of arc; (f) 1000/r.

P. 233–234. For r = 10(5)80(10)100(25)200 are given angles at center in grades subtending arcs 1(1)10. This is followed (p. 235–239) by lengths of arcs of a unit circle opposite central angles: (a) [1ᵍ(1ᵍ)100ᵍ(5ᵍ)300ᵍ(10ᵍ)400ᵍ; 11D], (b) [1°(1°)100°(10°)270°(30°)360°; 11D], (c) [1′(1′)60′; 11D], (d) [1″(1″)60″; 11D]. Conversion of centesimal grades, minutes and seconds, and conversely (p. 241–247). Various trigonometric formulae and constants (p. 250–255).

R. C. A.

488[D, R].—Great Britain, H. M. Nautical Almanac Office, *Five-Figure Tables of Natural Trigonometrical Functions.* London, His Majesty's Stationery Office, 1947, iv, 124 p. 24.8 × 30.5 cm. Bound, 15 shillings. American agent: British Information Services, 30 Rockefeller Plaza, New York 20, N. Y. $4.00.

The present tables were prepared in 1941 by the Nautical Almanac Office, at the request of the War Office, for use in survey calculations; they were not put on general sale to the public either then or when they were reprinted in 1945. It has now been possible to make the tables, which are the only five-figure natural trigonometrical tables at such a small interval, available to the public.

These tables give natural values of the four trigonometrical functions that occur most frequently in surveying and associated problems, and are intended to replace similar tables (*Manual of Artillery Survey*, Part II, 1924) giving logarithmic values of the same four functions. They are provided specially for use with calculating machines and are not of great practical utility unless such a machine is available. The combination of natural trigonometrical functions and a calculating machine provides, however, a more powerful and effective weapon for the practical computation of survey calculations than the superseded combination of logarithmic values and tables; in particular, mechanization allows of a more direct approach, leading to considerable simplification of the formulae employed.

Many sources are available for the trigonometrical functions to five figures, so that it is difficult to quote an authority for the source of the present figures. They have, however, been rigorously compared in proof with (among other tables) Andoyer's 15-figure values, and these can therefore justly be regarded as the relevant source.

There are two separate tables, the "auxiliary" and the "main" tables.

The auxiliary table (p. 2–31) gives, directly, values of the cotangent for every second of arc up to 7°30′ to 5S. Each column gives the 60 values of the function in each minute of arc and there are 15 columns to the page, so that each degree occupies four pages or two openings.

The main table (p. 34–123) contains values of the sine, tangent, cotangent and cosine arranged semi-quadrantally at interval 10″. Five decimals are retained throughout in the sine, tangent and cosine, but the values of the cotangent have been given to 5S only in most of the range; from 27° to 45° 5D have been retained. Each block contains values of all four functions for 10′ and three blocks are given on each page, so that each opening of the book covers a whole degree; thus, the functions for a given sub-division of a degree are always

found in the same relative position in the opened book. In each of the blocks the signs of the functions in each of the quadrants are indicated.

The tables have been constructed at such a small interval of angle that, in field survey work, no interpolation between tabular entries is necessary. If for a special purpose more accurate values of the functions are required, the values of sine, tangent and cosine may be interpolated · by simple proportion, the maximum difference between consecutive entries being 10; for the cotangent the differences are larger, but simple proportional interpolation is permissible, except in the main table 0°0' to 0°17' and in the auxiliary table from 0'0" to 1'40".

Extracts from the text

EDITORIAL NOTE: The printing and display of the table are excellent. A list of errata in the 1941 and 1945 editions is given in the preface, signed by the Astronomer Royal, H. SPENCER JONES, April, 1947. The volume was published in July and a brief errata sheet, dated October, has been inserted.

489[F].—F. J. DUARTE, "Sobre la ecuacion $x_1^3 + x_2^3 = y_1^3 + y_2^3$," Acad. Ciencias Fis., Mat. y Nat., Caracas, Venezuela, *Boletín*, no. 23, 1943, 19 p.

This note on the famous Diophantine equation of EULER contains (p. 15–19) a list of 100 solutions in integers of this equation. These solutions are arranged in no definite order and this makes it difficult to compare the list with others. The bibliography of 12 titles should have included the authoritative article by RICHMOND[1] in which is given a list of 63 solutions in integers not exceeding 100 in absolute value. Of these only 25 are given by Duarte who, however, gives the solution 29, 99, 60, 92 overlooked by Richmond. The majority of solutions of Duarte have large and nearly equal values of x_2 and y_2. The largest solution is

$$309032, \qquad 390545, \qquad 313532, \qquad 387665.$$

These reflect the author's methods of solution rather than any general property of the class of all solutions.

D. H. L.

[1] H. W. RICHMOND. "On integers which satisfy the equation $t^3 \pm x^3 \pm y^3 \pm z^3 = 0$," Camb. Phil. Soc., *Trans.*, v. 22, no. xix, 1920, p. 389–403.

490[F].—A. FERRIER, *Les Nombres Premiers*, Paris, Vuibert, 1947, vi + 111 p. 14 × 22.5 cm. 280 francs.

This work contains (p. 60–110) a factor table to 100000 of all numbers not divisible by primes less than 17. With each such number which is not a prime is given its least prime factor. Primes are indicated by boldface type. No account of the construction or possible comparison of this table with others is given. In spot checking the table only two errata were discovered by the reviewer: p. 110, the final digits of 99199 should not be boldface whereas the final digits of 99989 should be boldface. This "type 5" table differs from any factor table previously published. The nearest similar table is due to Cahen[1] which omits primes and extends only to 10000.

Two other tables may be mentioned: p. 29, a table of 2^n and the factors of $2^n \pm 1$, for $n \leq 40$; p. 32, a table of the number of primes not exceeding x for $x = k \cdot 10^n$, $k = 1(1)10$, $n = 1(1)6$. The first half of this work consists of an interesting summary of various results and methods having to do with primes. To quote the author, "Mystérieux et inoffensifs, les nombres premiers offrent un refuge attrayant et serein à qui veut s'écarter de la civilisation de l'atome." The material presented is definitely pre-atomic. Twentieth century advances on the problem are, with a few exceptions, entirely omitted.

D. H. L.

[1] E. CAHEN, *Théorie des Nombres*, v. 1, Paris, 1914, p. 378–381.

491[F].—ALBERT GLODEN, (a) *Table des Solutions de la Congruence* $x^4 + 1 \equiv 0$ *(mod p) pour 350 000 $< p <$ 500 000.* Luxembourg, author, and Paris, Centre de Documentation Universitaire, 5 place de la Sorbonne, 1946, 41 p. 21.5 × 27.5 cm. Offset print, except title page. (b) *Table des Solutions de la Congruence* $x^4 + 1 \equiv 0$ *(mod p) pour 500 000 $< p <$ 600 000.* Luxembourg, author, rue Jean Jaurès 11, December 1947, i, 12 leaves. 20.8 × 29.5 cm. Offset print on one side of each leaf. For $p < 3.5 \cdot 10^5$ see *MTAC*, v. 2, p. 71, 210–211 (reviews by D. H. L.).

The titles of these publications and previous reviews are sufficiently descriptive of the contents of (a) and (b), forming the bases for factorizations; compare *MTAC*, v. 2, p. 72, 211, 300.

We have also received a carbon of a type-script table (dated Dec. 2, 1947) of single solutions of $x^4 + 1 \equiv 0$ (mod p), $p > 6 \cdot 10^5$, for 243 values of p (from 600 841 to 9 778 057) compiled by means of the author's factorizations of $x^4 + 1$.

In a communication of 29 November, 1947, "Nouveaux compléments aux tables de factorisations de Cunningham," sent for publication in *Mathesis*, Professor Gloden reported having shown, by means of (b), that the following 21 numbers are prime:

(i) $X^4 + 1$ for $X = 710, 730, 732, 738, 742, 748, 758, 760, 768, 772$;

(ii) $\frac{1}{2}(X^4 + 1)$ for $X = 843, 845, 855, 857, 879, 883, 891, 895, 913, 917, 919$.

He gives also the new factorizations

$$\frac{1}{2}(889^4 + 1) = 505\,777 \cdot 617\,473$$
$$38^8 + 1 = 539\,089 \cdot 8\,065\,073.$$

Compare *MTAC*, v. 2, p. 252.

492[F].—NILS PIPPING, "Tabel der Diagonalkettenbrüche für die Quadratwurzeln aus den natürlichen Zahlen von 1–500." Aabo, Finland, Akademi, *Acta, Mathem. et Phys.*, v. 15, no. 10, 1947, 11 p.

MINKOWSKI defined the real semi-regular continued fraction

$$\xi = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \quad (a_i^2 = 1)$$

to be a *diagonal continued fraction* in case each of its convergents An/Bn differs from ξ by an amount less than $1/(2B_n^2)$ in absolute value. Every real number not equal to half an integer has precisely one such expansion. The present note contains a table of the diagonal continued fraction expansions of all square roots of non-square integers $D \leq 500$. These are periodic and so it suffices to give the period together with the nonperiodic term $b_0 = [D^{\frac{1}{2}}]$. In case $a_k = -1$, b_k is printed with a prime to indicate this fact. For example for $D = 183$ the entry is 183/14 (2', 8, 2, 28'). This means that

$$183^{\frac{1}{2}} = 14 - \frac{1}{2} + \frac{1}{8} + \frac{1}{2} - \frac{1}{28} - \frac{1}{2} + \frac{1}{8} + \cdots.$$

In 144 of the 478 expansions the diagonal continued fraction is regular, i.e., the a's are all equal to unity.

<div align="right">D. H. L.</div>

493[F].—DOV YARDEN, "Hashlamoth leluah mispare Fibonatsi" [Addenda to the table of Fibonacci numbers], *Riveon Lematematika*, v. 2, Sept. 1947, p. 22. 21.6 × 33.6 cm. See also MTE 127.

These are the addenda promised in connection with a previous table of the factors of Fibonacci's U_n and $V_n = U_{2n}/U_n$ (see *MTAC*, v. 2, p. 343). New factors are given of U_n

for $n = 67, 115$ and of V_n for $n = 94, 97, 98, 101, 103, 104, 106, 108, 114, 119, 122, 127$. The large factors of uncertain character of the following numbers have no factor under 10^5.

U_n for $n = 79, 91, 93, 111, 115, 125.$

V_n for $n = 73, 74, 79, 83, 86, 91, 98, 103, 106, 108, 110, 117, 119, 126.$

D. H. L.

494[G].—A. ZAVROTSKY, "Algunas generalizaciones del concepto de campo," Acad. Ciencias Fis. Mat. y Nat., Caracas, Venezuela, *Boletín*, no. 28, año 1946, 1947, 23 p.

This article contains tables of the functions $S_2(x, y; n)$ defined as follows:

$$S_2(x, y; -1) = \log_2 (2^x + 2^y), \qquad S_2(x, y; 0) = x + y,$$
$$S_2(x, y; 1) = xy, \qquad S_2(x, y; 2) = x^{\log_2 y},$$
$$S_2(x, y; 3) = 2^{(\log_2 x)^u}, \qquad u = \log_2 \log_2 y.$$

In general if we denote by $l_n(x)$ the nth iterated logarithm of x to base 2 so that

$$l_{-1}(x) = 2^x, \qquad l_0(x) = x, \qquad l_1(x) = \log_2 x,$$
$$l_2(x) = l[l(x)] = \log_2 \log_2 x, \cdots,$$

then

$$S(x, y; n) = l_{-n}[l_n(x) + l_n(y)].$$

The tables are for $x = 0(1)10$, $y = 0(1)10$, $n = -1, 2, 3$, and are to 5D for $n = -1$, and 3D otherwise.

These same functions may be defined for the non-zero elements in $GF(p)$ and then they become periodic functions of n. This is illustrated by a set of tables for $GF(5)$.

D. H. L.

495[I].—H. E. SALZER, "Tables for facilitating the use of Chebyshev's quadrature formula," *Jn. Math. Physics*, v. 26, 1947, p. 191–194. 17.5×25.4 cm.

CHEBYSHEV suggested in 1874 the quadrature formula[1]

$$\int_{-1}^{1} f(z)dz = 2n^{-1} \sum_{i=1}^{n} f(z_i) + R_n$$

whose coefficients are all equal. The remainder R_n may be made zero for f any polynomial of degree n provided the points z_i are taken as the roots of a polynomial $T_n(x)$ of degree n which is the polynomial part of the function

$$x^n \exp \left\{ -n \sum_{k=1}^{\infty} (2k[2k+1]x^{2k})^{-1} \right\}.$$

This paper gives the first twelve polynomials. By a curious stroke of misfortune the roots of $T_n(x)$ are all real only for $n = 1(1)7, 9$, so that Chebyshev's idea is practical in these cases only. The problem of checking the smoothness of the computed values $f(z_i)$ is solved by expressing the divided difference of order $n - 1$ of $f(z)$ as a linear combination of these n values. The requisite coefficients are tabulated, mostly to 9S, together with the roots z_i to 10D, for $n = 3(1)7, 9$. For a similar treatment of the Gaussian quadrature formula see *MTAC*, v. 2, p. 256.

D. H. L.

[1] P. L. CHEBYSHEV, (a) "Sur les quadratures," *Jn. de Math.*, s. 2, v. 19, 1874, p. 19–34; (b) Assoc. Française pour l'Avanc. d. Sci., *Compte Rendu*, Lyon, 1873, p. 69–82; (c) *Oeuvres*, ed. by A. MARKOV & N. SONIN. St. Petersburg, v. 2, 1907, p. 165–180. Chebyshev's formula was suggested by a formula due to B. BRONWIN, *Phil. Mag.*, v. 34, 1849, p. 262.

EDITORIAL NOTES: The statement of this article which Mr. Salzer modified in referring to Walther's table (1930), MTE 126, must also be slightly modified in connection with Chebyshev's table (1873) of the roots of $T_n(z) = 0$, $n = [2(1)7; 6D]$, (a) p. 25–26, (b) p. 74–75, (c) p. 170–171. Comparison with the Salzer table shows the following errors in the original table; (a), (b): $n = 2$, *for* .816479, *read* .577350; $n = 3$, *for* .707166, *read* .707107; $n = 4$, *for* .794622 and .187597, *read respectively* .794654 and .187592; $n = 5$, *for* .832437 and

.374542, *read respectively* .832497 and .374541; $n = 6$, *for* .866249, .422540, and .266603, *read respectively* .866247, .422591, and .266635; $n = 7$, *for* .883854, .529706 and .323850, *read respectively* .883862, .529657 and .323912. The two errors on p. 159 of E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, London, 1924, are evidently due to copying Chebyshev's erroneous values for $n = 5$. All of Chebyshev's errors have been corrected in (c).

496[K].—TRUMAN LEE KELLEY, *Fundamentals of Statistics*, Cambridge, Harvard University Press, 1947, xvi, 755 p. 14.5 × 22.1 cm. $10.00. Compare *MTAC*, v. 1, p. 151–152.

The word "statistics" covers many different meanings. Originally the work of the statistician was supposed to consist in collecting, processing, and presenting empirical data. The modern theoretical statistician lives in a different world. The inadequacy of the above description is seen from the fact that in the theory of designs of experiments (as applied in agriculture and industrial experimentation) and in the design of efficient sampling techniques the mathematical statistician deals with data which do not yet exist. He is concerned with abstract models and theorems much in the same way as the theoretical physicist or engineer. Now even the simplest statistical techniques for analyzing empirical data or for judging reliability and significance of apparent differences are based on rather intricate mathematical theories. Clearly the majority of users of such techniques have not the necessary background for a proper understanding of statistical theory. It is therefore necessary to present the result of modern statistical investigation in a descriptive way and give rules rather than proofs. This corresponds to the education of the engineers who use formulae and rules without knowledge of the underlying physical theories.

The present book follows this pattern. It is intended for people without any mathematical background and is to serve as a first introduction into a rational analysis of quantitative data. Accordingly, the first five chapters (198 pages) are devoted to the general cultural background and to a description of various techniques of graphical and tabular presentation. These are a prerequisite in the same sense as arithmetic. The following three chapters (pages 199–310) are devoted to classical descriptive statistics: moments, means, normal distribution, etc. Statistical methods in the proper sense of the word begin only with the ninth chapter. The choice of topics is obviously made with a view to users in the fields of education and psychology. The main subjects discussed are regression and correlation analysis and some chi-square analysis. Various elementary mathematical techniques are described, such as interpolation, curve-fitting, solution of linear algebraic equations, etc. Chapter 13 also describes the fundamental idea underlying the sequential analysis of ABRAHAM WALD.

The following mathematical tables are contained in the book. On pages 628–636 three-point Lagrange interpolation coefficients for $[0(.001)1; 5D]$; on pages 637–638 four-point interpolation coefficients $[0(.01)1; 5D]$. On pages 640–652 tables of the error function are arranged as follows: Put $z = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$, $I = \int e^z z dx$. The argument of the tables is $I = 0(.001).5$ (this covers the entire range). The successive columns give x and z to 6D each; then $q = .5 - I$, followed by z/q (to 5D), z/p (to 5D), pq and $p = 1 - q = .5 + I$ (the latter exact). On pages 653–656 we find the square roots of N and $10N$, and the cube roots of N, $10N$, $100N$ for $N = [1(.1)10; 5D]$ together with indications of the maximum error committed when interpolating. In the text there is on p. 587 a table of $\Gamma(x)$ for $x = [.99(.01)2.01; 5D]$ and on p. 596 of $x = 2 \sin^{-1} p^{\frac{1}{2}} - \frac{1}{2}\pi$ for $p = [0(.01)1; 4D]$. None of the tables offers anything new.

Cornell University   WILL FELLER

497[K, P].—CONNY C. R. A. PALM, *Table of the Erlang Loss Formula.* (*Tables of Telephone Traffic Formulae*, no. 1.) Printed in Göteborg, Erlanders Boktryckeri, 1947; distributed by C. E. Fritzes Hovbok-handel, Stockholm, Sweden, ii, 23 p. 21.3 × 28 cm.

The simplest situation in the theory of automatic telephone exchanges can be described as follows. Incoming calls are "randomly distributed" (that is, at least during the "busy

hour"). The probability that exactly k calls will originate during time t is given by the POISSON expression $e^{-at}(at)^k/k!$ where $a > 0$ is a constant. Moreover, it is assumed that the lengths of the individual conversations are statistically independent and that the probability of any conversation lasting for time t or more is $e^{-(t/b)}$. Suppose there are n circuits available so that an incoming call is served whenever a circuit is free. If no circuit is free, the call is "lost" and has no influence on the future traffic. Erlang's formula[1] for the proportion of lost calls is

$$E_{1,n}(A) = A_n(A_0 + A_1 + \cdots + A_n)^{-1}, \ A = ab,$$

where we put for abbreviation $A_k = (ab)^k/k!$ (the notation is not standard and the general conditions of applicability of Erlang's formula are still discussed). The tables are double-entry 6D tables with $n = 1(1)150$, and $A = .05(.05)1(.1)20(.5)30(1)50, \ 52(4)100$. Only entries which are significantly different from zero are tabulated, and thus the effective range of A decreases with increasing n. For $n = 100$ entries occur only for $A \geqslant 56$, and for $n = 150$ we have the single entry $E_{1,150}(100) = .000 001$. The tabular intervals are selected so as to permit linear interpolation in A with an accuracy to four or five decimals. The sixth decimals are said to be accurate throughout the tables. These have been computed from the recurrence formula $E_{1,n}(A)\{n + AE_{1,n-1}(A)\} = AE_{1,n-1}(A)$.

<div align="right">WILL FELLER</div>

Cornell University

[1] AGNER K. ERLANG, (a) "Løsning af nogle Problemer fra Sandsynlighedsregningen af Betydning for de automatiske Telefoncentraler, *Elektroteknikeren*, Copenhagen, v. 13, 1917, p. 5–13; (b) "Lösung einiger Probleme der Wahrscheinlichkeitsrechnung von Bedeutung für die selbsttätigen Fernsprechämter," *Elektrotechn. Z.*, v. 39, 1918, p. 504–508, with tables, translation of (a). Also T. C. FRY, *Probability and its Engineering Uses*, New York, 1928, p. 342–347.

498[L].—JOHN P. BLEWETT, "Magnetic field configurations due to air core coils," *Jn. Appl. Physics*, v. 18, no. 11, Nov. 1947, p. 968–976; tables p. 969–970. 20 × 26.5 cm.

The field configurations around a circular loop of wire bearing current are discussed, and a tabulation is presented for the field component parallel to the axis of the loop. The table gives 4D values of the function $B_z a/uI$, for $z/a = 0(.02).36$, and $\rho/a = 0(.1).5(.02)$-1.5(.1)2.

$$B_z a/uI = 2a[(a + \rho)^2 + z^2]^{-\frac{1}{2}}[K + E(a^2 - \rho^2 - z^2)/\{(a - \rho)^2 + z^2\}],$$

K and E being the complete elliptic integrals of the modulus $k^2 = 4a\rho/[(a + \rho)^2 + z^2]$.

499[L].—CHRISTOFFEL JACOB BOUWKAMP, **A.** *Theoretische en Numerieke Behandeling van de Buiging door een Ronde Opening* [Theoretical and numerical treatment of diffraction by a circular aperture]. Diss. Groningen. Groningen, Holland, J. B. Wolters' U. M., 1941, vi, 60, 3 p. 16 × 24.5 cm. **B.** "On spheroidal wave functions of order zero," *Jn. Math. Physics*, v. 26, July 1947, p. 79–92. 17.4 × 25.3 cm.

The differential equation for spheroidal wave functions of order zero is written in the form

$$(1.1) \qquad d[(1 - \xi^2)dX/d\xi]/d\xi + (k^2\xi^2 + \Lambda)X = 0,$$

"where k is a given parameter and Λ is one of a set of characteristic numbers $\Lambda_0, \Lambda_1, \cdots, \Lambda_m$ such that for $\Lambda = \Lambda_m$, (1.1) has one and only one solution $X_m(\xi)$ which is an integral function of ξ." The function X_m can be expressed by

$$X_m(\xi) = \sum_{n=0}^{\infty} b_{n,p}^{(m)} P_{2n+p}(\xi),$$

where $p = 0$ if m is even, $p = 1$ if m is odd, and $P_{2n+p}(\xi)$ is the LEGENDRE polynomial of

degree $2n + p$. The coefficients $b_{2n+p}^{(m)}$ are so normalized that

$$\int_{-1}^{1} X_m{}^2(\xi)d\xi = \int_{-1}^{1} P_m{}^2(\xi)d\xi = 2/(2m + 1),$$

and $b_m \to 1$ for $k = 0$.

Some of the tabular material and formulae relating to spheroidal wave functions, given in **A**, are also in **B** in more complete form. We shall indicate by an asterisk all material of **A** reproduced or elaborated upon in **B**.

The following tables are given in **A**:

(1)* $\Lambda_0, \Lambda_1, \cdots, \Lambda_6$ and coefficients b_n for $k^2 = 3(1)10$; 6D.

(2) $\Lambda_0, \Lambda_1, \cdots, \Lambda_6$ and coefficients b_n for $k^2 = 15, 20, 25, 50, 100$; 6D.
        $\Lambda_8, \Lambda_{10}, \Lambda_{12}$ and coefficients b_n for $k^2 = 25$; 6D.
        $\Lambda_7, \Lambda_8, \Lambda_9, \Lambda_{10}$ and coefficients b_n for $k^2 = 100$; 6D.

(3) $X_{2m}(0), X_{2m}(1), X_{2m+1}(1), X'_{2m+1}(0)$ determined by the coefficients of (1) and (2) above.

(4)   Values of σ_n and $2|\sigma_n|^2/(2n + 1)$ (σ_n complex), for $k^2 = 3(1)10$; 15, 25, 50, 100, and values of n, up to point where $|\sigma_n|^2$ vanishes, to 4D. Let

$$\varphi = (1/r) \exp(- ikr)\sum_0^\infty \sigma_{2n+p}X_{2n+p}(\cos \vartheta), \quad p = 0 \text{ or } 1;$$

then "for the radiation of sound the behavior of the potential at large distances from the aperture" is found to be φ.

The following formulae contained in this paper should be of interest from a computational viewpoint:

(a)* $\rho_0, \rho_1, \rho_2, \rho_3$ of the formula $\Lambda_m = \sum_{r=0}^{\infty} \rho_r k^{2r}$, for general m; also *numerical* values of $\rho_2, \rho_3,$
      $m = 0(1)12$; 6D in ρ_2. 9D in ρ_3.

(b) Expressions for b_{m+2r}, $\nu = 0, 1, 2, 3$ for general m, as a power series in k^2 up to k^6;

(c)* Power series for Λ_0, Λ_2 to terms through k^{10}; for Λ_4, Λ_6, to terms through k^8. Also, of coefficients $b_0, b_2, \cdots, b_8$ associated with Λ_0 to k^8; of $b_0, \cdots, b_8$ associated with Λ_2 to k^6.

(d)* Power series for $X_0(0), X_0(1)$, through k^8; of $X_1'(0)$ through k^6, all coefficients in both fractional and decimal (8D) form.

(e) Asymptotic expansion of Λ_m, general m, in powers of $1/k$, through the power $1/k^2$.

(f) It is shown how to construct a second solution of (1), $Y_m(\xi)$; an error of STRUTT[1] in regard to this solution is noted.

(g)* Method of constructing Λ_m and the coefficients b_n from the continued fractions associated with the characteristic values, and of improving the approximation by Newton's method, applied at a judiciously chosen stage of the computations.

(h) Some graphs and other short tables of a specialized nature, mainly relating to the physical problem.

(A number of changes in the tables and formulae appear to have been made after publication, but it may be that all copies in circulation have been similarly corrected. There is a notation that of the six decimal places given in the tables, roughly four are correct.)

In the paper **B**, the following tables are given:

(5) $\Lambda_0, \Lambda_1, \Lambda_2$ for $k^2 = - 10(1)10$; 6D.

(6) $\Lambda_3, \cdots, \Lambda_6$ for $k^2 = 0(1)10$; 6D.

(7) Coefficients b_n associated with $X_0(\xi)$ and $X_1(\xi)$, $k^2 = - 10(1)10$; 6D.

(8) Coefficients b_n associated with $X_2(\xi), \cdots, X_6(\xi)$, $k^2 = 0(1)10$; 6D.

Formulae are given in **B** for the following:

(i) Power series for Λ_0, Λ_1 through term involving k^{12}; for Λ_2 through k^{10}; for $\Lambda_3, \cdots, \Lambda_6$ through k^8. All these coefficients of the series are given in both fractional and decimal form, the latter to 12D.

(j) Power series for b_n associated with Λ_0, Λ_1, through k^8; for b_n associated with Λ_2, through k^9, n either odd or even, $\leqslant 9$.

(k) Power series from Λ_m, m general, through term involving k^5.

(l) Details of the method described in (g)* and a list of 31 bibliographic references.

In connection with (i), Bouwkamp mentions a paper by SANDEMAN[2] giving terms for Λ_0 through k^{14}. Although the formulae given in **A** and **B** are not all new, the author has noted some errors made by earlier writers, and the accessibility of **B** makes it a welcome source for the known results.

RELATION TO FUNCTIONS TABULATED ELSEWHERE. The most extensive tables now in existence are probably those contained in J. A. STRATTON, P. M. MORSE, L. J. CHU, and R. A. HUTNER, *Elliptic Cylinder and Spheroidal Wave Functions*, 1941 (see *MTAC*, v. 1, p. 157–160). In this work (later to be designated by S) the notation is as follows:

For Prolate Spheroidal Functions

$$(2.1) \qquad d[(\eta^2 - 1)dS/d\eta]/d\eta + [A + c^2\eta^2 - m^2/(\eta^2 - 1)]S = 0.$$

For Oblate Spheroidal Functions

$$(2.2) \qquad d[(\eta^2 - 1)dS/d\eta]/d\eta + [B - c^2\eta^2 - m^2/(\eta^2 - 1)]S = 0, \qquad 0 \leqslant \eta \leqslant 1.$$

$$(2.3) \qquad d[(\xi^2 + 1)dR/d\xi]/d\xi + [B + c^2\xi^2 + m^2/(\xi^2 + 1)]R = 0, \qquad |\xi| > 1.$$

(2.2) is obtainable from (2.1) by replacing c by ic.

The *characteristic values* corresponding to a given c and fixed integer m are denoted by $A_{m,l}$ or $B_{m,l}$. The relations between the functions of S and Bouwkamp are therefore as follows:

S Notation	c	η	$B_{0,l}$
Bouwkamp Notation	k	ξ	$-\Lambda_m$

It should be emphasized that "m" of Bouwkamp corresponds to "l" of S, and that Bouwkamp deals *only with functions of order zero*. Where Bouwkamp tabulates results for $-k^2$, they correspond to the *prolate* spheroidal functions and positive k^2; results for positive k^2 corresponding to *oblate* spheroidal functions. Functions tabulated in S: $A_{m,l}$, $B_{m,l}$ and coefficients of the series to about 5 significant figures—to be denoted by (m, l): $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$, $(3, 0)$, for $c = 0(.2)5$ and $.1$; $.5(.5)4.5$, for both the *oblate* and *prolate* cases. The range of the S functions for $m = 0$ and $l = 0, 1, 2, 3$ is therefore larger than that in Bouwkamp (except for the isolated values corresponding to $k^2 = 50,100$). However, since Bouwkamp's parameter is k^2, i.e., c^2 of S—there is little overlapping in results. For $l > 3$, Bouwkamp's results apparently are not duplicated elsewhere except for some unpublished results noted later.

The normalization in S is different from that of Bouwkamp.

Mr. FRED LEITNER of the Applied Mathematics Group, New York University, has the following unpublished results:

$B_{m,l}$ for $c = 1(1)5$; $m = 0$, $l = 4, 5$; $m = 1$, $l = 3, 4, 5, 6, 7$;
$m = 2$, $l = 0, 2, 3, 4, 5, 6, 7$; $m = 3$, $l = 1(1)7$; $m = 4$, $l = 0(1)8$;

with coefficients of another type of series (used by LEIGH PAGE) and values of functions. The characteristic values for $m = 0$, $l = 4, 5$ therefore duplicate those of Bouwkamp, but the rest are mostly new.

An interesting formula for obtaining $B_{m,l}$ has recently been communicated to the NBSCL by Dr. WM. F. EBERLEIN of the Institute for Advanced Study. This formula is in reciprocal powers of c (through c^{-8}) for *general values* of m and l. It is therefore of wider application than the one given in Bouwkamp. This formula will probably be published by its author at some future date.

GERTRUDE BLANCH

NBSCL

[1] M. J. O. STRUTT, *Phys. Z.*, v. 69, 1931, p. 597, and *Lamésche, Mathieusche und verwandte Funktionen in Physik u. Math.*, *Erg. d. Math.*, v. 1, no. 3, 1932.
[2] IAN SANDEMAN, R. Soc. Edinburgh, *Proc.*, v. 55, 1935, p. 77.

500[L].—J. G. CHARNEY, "The dynamics of long waves in a baroclinic westerly current," *Jn. of Meteorology*, v. 4, Oct. 1947, p. 135–162; tables p. 160–162. 21.5 × 27.8 cm.

T.1 gives 4S (a few 5S) values of the function $\psi_1(\xi, r)$ satisfying the confluent hypergeometric equation

(1) $$\xi\psi'' - \xi\psi' + r\psi = 0, \text{ for } \xi = 0(.1).2(.2)1(1)10,$$

$r = -1.5, -0.5, .1(.1).9, 1.5(1)5.5$ **T.2** gives 3D values of the function $\psi_1' = d\psi_1/d\xi$ for the same values of ξ and r; for $r > .8$ the values are given to 4 or 5S. $\psi_1(\xi, r) = \dfrac{\sin \pi a}{\pi}$

$$\times \left\{ a\xi M(a + 1, 2, \xi) \left[\ln \xi + \frac{\Gamma'(a)}{\Gamma(a)} - 2\frac{\Gamma'(1)}{\Gamma(1)} \right] + 1 + \sum_{n=1}^{\infty} B_n \frac{a(a+1)\cdots(a+n-1)}{(n-1)!n!} \xi^n \right\};$$

$$B_n = \sum_{\nu=0}^{n-1} \left(\frac{1}{a+\nu} - \frac{2}{1+\nu} \right) + \frac{1}{n}; a = -r.$$

T.3 gives values of $\psi_2(\xi, r) = \xi M(a + 1, 2, \xi)$, where $a = -r$, and $M(a, b, \xi) = 1 + \dfrac{a}{1!b}\xi + \dfrac{a(a+1)}{2!b(b+1)}\xi^2 + \cdots$. The function ψ_2 also satisfies (1). This table, mostly 5D, is for $r = .1(.1).9, 1.5, 2.5,$ and $\xi = 0(.1).2(.2)1(1)8$; the values of $\xi = 5(1)8$ being given only for $r > .8$. **T.4** gives 3D values of $\psi_2' = d\psi_2/d\xi$, for $r = .7(.1).9, \xi = 0(.1).2(.2)1(1)4$. **T.5** gives 3 or 4S values of $X = \xi[\ln \xi + \sum_{n=1}^{\infty} \xi^n/(n+1)!n] - 1$, and its derivative, for $\xi = 0(.1)-.2(.2)1(1)7$.

501[L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 7, *Tables of Bessel Functions of the First Kind of Orders Ten, Eleven, and Twelve*; v. 8, *Tables of Bessel Functions of the First Kind of Orders Thirteen, Fourteen, and Fifteen*. By the staff of the Laboratory, Professor H. H. Aiken, technical director. Cambridge, Mass., Harvard University Press, 1947, [x, 636] p., [x, 614] p. 19.5 × 26.7 cm. $10.00 + $10.00. Offset print. Compare *MTAC*, v. 2, p. 176f, 185f, 261f, 344.

These two new volumes of the magnificent series eventually to tabulate $J_n(x)$, $n = 0(1)100$, are 10D tables for $x = 0(.001)25(.01)99.99$, and $n = 10(1)15$. The first significant values, .00000 00001, are for $J_{10}(.847)$, $J_{11}(1.140)$, $J_{12}(1.471)$, $J_{13}(1.837)$, $J_{14}(2.236)$, $J_{15}(2.663)$.

An incidental use for such tables is to determine the 3D or 2D values (<100) of Bessel function zeros; for example, the first 25 zeros of $J_{15}(x)$ are (unrounded) as follows: 17.801, 21.956, 25.70, 29.27, 32.73, 36.12, 39.46, 42.78, 46.06, 49.33, 52.57, 55.81, 59.04, 62.25, 65.46, 68.66, 71.86, 75.05, 78.24, 81.42, 84.60, 87.78, 90.96, 94.13, 97.30.

Since $J_{14}(18.899) = 0.00015$ 08840 and $J_{14}(18.900) = -0.00000$ 03095, the first zero of $J_{14}(x)$ is, to 3D (unrounded), 18.899. The correctly rounded value 18.9000 is given by D. B. SMITH, L. M. RODGERS & E. H. TRAUB (1944, see *MTAC*, v. 2, p. 48–49).

The results in these volumes are almost entirely new. Among published tables to at least 10D, for $n = 10(1)15$, are only those of MEISSEL (1895) for $x = [0(1)24; 18D]$; and of HAYASHI (1930) for $x = 1, 2, 10(10)50$, to at least 15D. For $n = 10(1)13$, AIREY (1915) gave a table for $x = [6.5(.5)16; 10D]$. For various differences between Airey and Harvard see MTE 124.

R. C. A.

502[L].—H. KOBER, *Dictionary of Conformal Representations.* Admiralty, Department of Physical Research, Mathematical and Statistical Section. Part III. *Exponential Functions and some related Functions,* $[w = e^z,$ $w = \ln z$, and simple combinations of these functions with functions discussed in earlier parts], iv, 52 leaves. Part IV, *Schwarz-Christoffel transformations representable in terms of elementary transformations,* iii, ix, 17 leaves. Numbers SRE/ACS 109 and 110 = ACIL/ADM 47/562 and 47/818 [ACIL = Admiralty Center for Scientific Information and Liaison]. London, 1947. This publication is not available for general distribution. 20.2 × 33.1 cm. The author is HERMANN KOBER (1888–), Ph.D., Univ. Breslau, 1910.

We have already reported on the publication of the first two parts in *MTAC*, v. 2, p. 296–297. Part V, *Higher Transcendental Functions,* is yet to appear. We are told in Part IV that "The reference to Admiralty Computing Service, Department of Scientific Research and Experiment, on the cover of this report has been used for the sake of conforming with earlier parts of the Dictionary. Correspondence relating to this and other Reports 'SRE/ACS' should be addressed to: Mathematical and Statistical Section, Department of Physical Research, Admiralty, Fanum House, Leicester Square, London W.C.2, which is a part of the Royal Naval Scientific Service and has superseded the earlier Organization."

503[L].—WILHELM MAGNUS & FRITZ OBERHETTINGER, *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete,* v. 52). Berlin, Springer, 1943, viii, 172 p. At the time of publication 13.20 marks, bound. 15.7 × 22.7 cm.

To give an adequate idea of the contents of this valuable non-numerical tabular work it seems desirable to transcribe the following contents of the nine chapters:

I(1–7): *Die Gammafunktion.*

II(7–16): *Die hypergeometrische Funktion*: 1. Die hypergeometrische Reihe; 2. Die Riemannsche Differentialgleichung.

III(16–48): *Die Zylinderfunktionen*: 1. Definitionen, Differentialgleichung, Rekursionsformeln, Reihenentwicklungen, Mehrdeutigkeit, unbestimmte Integrale; 2. Additionstheoreme, Multiplikationstheorem; 3. Asymptotische Entwicklungen, Multiplikationstheorem; 4. Nullstellen, Produktzerlegung für $J_{\nu}(z)$, Eine Partialbruchzerlegung; 5. Integraldarstellungen; 6. Integralbeziehungen zwischen Zylinderfunktionen; 7. Bestimmte Integrale mit Zylinderfunktionen, insbesondere diskontinuierliche Faktoren und Integraldarstellungen elementarer Funktionen; 8. Den BESSELschen Funktionen zugeordnete Polynome; 9. Die Funktionen von STRUVE, ANGER, und WEBER; 10. Die Funktionen von LOMMEL; 11. Beispiele KAPTEYNscher Reihen; 12. SCHLÖMILCH-Reihen; 13. MATHIEUsche Funktionen.

IV(49–78): *Kugelfunktionen*: 1. Differentialgleichung, Definitionen und Bezeichnungen; 2. Die LEGENDREschen Polynome; 3. Die zugeordnete LEGENDREsche Differentialgleichung erster Art; 4. Die Lösungen der LEGENDREschen Differentialgleichung; 5. Allgemeine Kugelfunktionen [(a). Darstellung durch hypergeometrische Funktionen; (b). Rekursionsformeln und Beziehungen zwischen verschiedenen Kugelfunktionen; (c). Formeln für spezielle Werte von x, μ, ν; (d). Analytische Fortsetzung und Verhalten für $|z| \geqslant 1$; (e). Integraldarstellungen; (f). Einige Integrale mit Kugelfunktionen; (g). Das Additionstheorem; (h). Sätze über Nullstellen; (i). Asymptotisches Verhalten für grosse Werte von $|\nu|$; (k). Ergänzungen]; 6. Kegelfunktionen; 7. Ringoder Torusfunktionen; [8.] Die Funktionen von GEGENBAUER.

V(78–86): *Orthogonale Polynome*: 1. TSCHEBYSCHEFFsche Polynome; 2. HERMITEsche Polynome; 3. JACOBIsche Polynome; 4. LAGUERREsche Polynome.

VI(86–98): *Die konfluente hypergeometrische Funktion und ihre Spezialfälle*: 1. Die Funktionen von KUMMER; 2. Die Funktionen von WHITTAKER; 3. Die Funktionen des parabolischen Zylinders; 4. Übersicht über die Spezialfälle der konfluenten hypergeometrischen Funktion [(a). LAGUERRESche Funktionen, (b). Die Funktionen des parabolischen Zylinders, (c). Die Zylinderfunktionen, (d). Die unvollständige Gammafunktion, (e). Das Fehlerintegral und die FRESNELschen Integrale, (f). Integrallogarithmus, Exponentialintegral, Integralsinus, Integralcosinus].

VII(98–114): *Thetafunktionen, elliptische Funktionen und Integrale*: 1. Thetafunktionen; 2. Die WEIERSTRASSsche $\wp$-Funktion; 3. Die elliptischen Funktionen von JACOBI; 4. Elliptische Integrale.

VIII(114–143): *Integraltransformationen und Integralumkehrungen*: 1. Die FOURIER-Transformation; 2. Die LAPLACE-Transformation; 3. Die HANKEL-Transformation; 4. Beispiele zur MELLIN-Transformation; 5. Über die GAUSS-Transformation; 6. Verschiedene Beispiele von Integralgleichungen erster Art [(i). Die Reziprozitätsformel von HILBERT für den Cotangens-Kern, (ii). Modifikationen der Formel von HILBERT, (iii). Die ABELschen Integralgleichungen, (iv). Integralumkehrungen vom Typ der MELLIN-Transformation, (v). Weitere Beispiele].

IX(144–161): *Koordinaten-Transformationen*. 1. Differentialoperationen in orthogonalen Koordinaten; 2. Beispiele zur Trennung der Veränderlichen; 3. Lineare Differentialgleichungen 2. Ordnung.

Then there are also: Collection of abbreviations used (p. 162–163), List of function symbols (p. 164–166), Literature list (p. 167–170), Subject and name list (p. 171–172). LAMÉ functions are not considered and MATHIEU functions little more than mentioned.

The copy of this work which we were permitted to see contained the following corrections by the authors: P. 1, l. 21, *for* $(\Gamma(\tfrac{1}{4})^4)$, *read* $(\Gamma(\tfrac{1}{4}))^4$; p. 3, l. 4, *for* $\sum_{n=1}^{\infty}\left(\dfrac{1}{n}, \textit{read } \sum_{n=0}^{\infty}\left(\dfrac{1}{n+1}\right.\right.$;

p. 4, l. 19, *for* $\binom{n+m-1}{n-1} = \binom{n+m-1}{m-1}$, *read* $m\binom{n+m-1}{n-1} = n\binom{n+m-1}{m-1}$; p. 5, l. 5, *for* $1 - y$, *read* $y - 1$; p. 9, l. 3, *for* $a - b$, *read* $b - a$, and *for* $a - c$, *read* $c - a$; l. 4, *for* $b - a$, *read* $a - b$, and *for* $c - b$, *read* $c - b$; p. 17, l. −5, *for* $e^{-ix\tau}$, *read* $e^{ix\tau}$; p. 18, l. −6, *for* $(-t)^n$, *read* $(-t)^{-n}$; p. 20, l. 7, *for* $\nu > 0$, *read* $r > 0$; p. 21, l. 4, *for* $J_n(kr) \cos \phi$, *read* $J_n(kr) \cos n\phi$; p. 22, l. −9, *for* $(\nu, 2m)$, *read* $(\nu, 2m + 1)$; p. 23, l. 10, *for* $\cos \alpha$, *read* $\sin \alpha$; l. −9 and −5, *for* $H_\nu^{(2)}(z)$, *read* $H_\nu^{(1)}(z)$; l. −5, *for* $3^{\frac{1}{2}}$, *read* $3^{-\frac{1}{2}}$ and *for* $\Gamma(\tfrac{1}{3})$, *read* $\Gamma(\tfrac{2}{3})$; p. 24, l. 10, *for* Es sei x reell, *read* Es seien x und ν reell; l. 11, *delete*; l. 14 and 15, *for* $e^{-i\nu}$, *read* $e^{i\nu}$, and *for* $\tfrac{1}{4}[-2\tau]^{\frac{1}{2}}$, *read* $\dfrac{\nu}{3} w^3$; l. −11, *delete* $-\dfrac{\pi}{2} \leqq \arg(-2\tau)^{\frac{1}{2}} < \dfrac{\pi}{2}$; l. −9, *delete* für reelle

Werte von τ; p. 26, l. 3, *for* 4z, *read* 4, and *delete* $j_{\nu, n}$; p. 33, l. −6, *for* $\ln \dfrac{a + \sqrt{a^2 + b^2}}{b}$ *read*

$\ln \dfrac{a + \sqrt{a^2 + b^2}}{b} \cdot \left(\dfrac{-2}{\pi}\right)$; p. 41, l. 10, *for* 3, *read* 3^2; p. 44, l. 7, between terms of summation *for* −, *read* +; p. 46, l. 13, for $(\lambda - 2h^2 \cos 2x) = 0$, *read* $(\lambda - 2h^2 \cos 2x)y = 0$; p. 49, l. 13, *for* anderrseits, *read* andrerseits; l. −12, *for* nubeschränkt, *read* unbeschränkt; p. 51, l. 8, *for* +5, *read* +3; p. 54, l. −5, *for* $P_n^{m'}(x)$, *read* $P_{n'}^{m}(x)$, and *for* $m \neq m'$, *read* $n \neq n'$; p. 61, l. −7, *for* $(2\nu + 1)$, *read* $-(2\nu + 1)$; p. 66, l. −9, *after* = *insert* $\pi^{\frac{1}{2}}$; p. 68, l. 6, *delete* $\sqrt{\pi}$; p. 71, l. 10, *for* $\dfrac{2}{\pi}$, *read* $\dfrac{\pi}{2}$, p. 73, l. 3, 5, *for* (Sin $\alpha)^{\mu/2}$, *read* (Sin $\alpha)^\mu$; l. 9, *for* −ν, *read* ν; p. 83, *after* l. 4, *insert* $-\gamma \neq 0, 1, \cdots, n - 1$; *after* l. 11, *insert* α, x reell, $\gamma > 0, \alpha > \gamma + 1$; p. 84, l. 9, *for* $\frac{1}{2}$, *read* $\frac{1}{4}$; p. 87, l. 3, *for* $c \neq 1, 2, 3, \cdots$, *read* $c \neq 0, \pm1, \pm2, \cdots$; p. 88, l. −1, *for* Zahlen ±1, *read* Zahlen 0, ±1; p. 89, l. −5, −7, *for* $W_{\mu, \kappa}$, *read* $W_{\kappa, \mu}$; p. 91, *for* $(z + t)^{-\frac{1}{2}-K-\mu}$, *read* $(z + t)^{-\frac{1}{2}+K-\mu}$; p. 92, l. −2, *for* $e^{-2i^3 - 2iss}$, *read* $e^{-2i^3 + 2iss}$; p. 95, l. −7, *for* $\frac{1}{2}$, *read* $\frac{1}{2}i$; p. 100, l. −1, *for* reell ist, *read* reell und irrational ist; p. 103, l. 6, *for* $e^{-\pi K/4K'}$, *read* $e^{-\pi K'/4K}$; p. 108, l. 3, *for* x (three times), *read* t; p. 114, l. 5, *for* $K/\sqrt{2}$, *read* $K\sqrt{2}$; l. 6, *for* $K/\sqrt{3}$, *read* $K\sqrt{3}$; l. 7, *for* tang, *read* tang^2; p. 119, in heading *for* §2, *read* §1, and l. 9, *for* b^2, *read* x^2; p. 125,

l. -5, *for* π, *read* $\sqrt{\pi}$; p. 131, l. -2, *for* a^{r+1}, *read* a^r; l. -1, Unterfunktion, *read* $\Gamma(\mu + \nu + 1)$ $\times (\mu^2 + a^2)^{-(\nu+1)/2} P_\nu^{-\mu}[\mu/(\rho^2 + a^2)^{\frac{1}{2}}] Re(\mu + \nu) > -1$; p. 132, l. 6, *for* p', *read* p^{r+1}; l. -4, *for* $J^2_{\nu+\frac{1}{2}}$, *read* $J_{r+\frac{1}{2}}$; l. -3, *for* K, *read* K_0; p. 139, l. -3, *for* $y - x$, *read* $z - y$; p. 156, l. 6, *for* $1/\sqrt{r}$, *read* $(2n+1)/\sqrt{r}$; l. 7, *for* $1/\sqrt{r_0}$, *read* $(2n+1)/\sqrt{r_0}$; p. 157, l. -2, *for* $(\xi^2 - \eta^2) = 0$, *read* $(\xi^2 - \eta^2)F = 0$; p. 158, l. -2, *insert* F before $=$; p. 161, *for* unabhängigen Lösungen, *read* unabhängigen reellen Lösungen.

<div align="right">R. C. A.</div>

504[L].—SAMUEL P. MORGAN, JR., *Tables of Bessel Functions of Imaginary Order and Imaginary Argument*. Pasadena, California Institute of Technology Book Store, 1947, vi, 61 leaves. 21.5 × 28 cm. Paper cover $2.75. Photographic reproduction, in an edition of 175 copies, of the original copy printed by IBM machines on one side of each leaf.

Various problems from different branches of mathematical physics give rise to the differential equation

$$(1) \qquad v^2 d^2 w/dv^2 + v dw/dv - (v^2 - \nu^2)w = 0,$$

in which v and ν are real quantities. Equation (1) is a special case of Bessel's equation,

$$(2) \qquad z^2 d^2 w/dz^2 + z dw/dz + (z^2 - \rho^2)w = 0,$$

in which $z = iv$, $\rho = i\nu$; and its solutions are therefore Bessel functions whose order and argument are both purely imaginary.

A fundamental real pair of solutions of (1) may be defined as follows:

$$F_\nu(v) = (\pi/\sinh \nu\pi)ReI_{i\nu}(v),$$
$$G_\nu(v) = -(\pi/\sinh \nu\pi)ImI_{i\nu}(v) = K_{i\nu}(v) = \tfrac{1}{2}\pi i e^{-\frac{1}{2}\nu\pi}H^{(1)}_{i\nu}(iv).$$

For brevity the functions $F_\nu(v)$ and $G_\nu(v)$ may be called "wedge functions" of the first and second kind respectively, since in potential theory they show a certain analogy to the solutions of Legendre's equation called "cone functions."

Representations of $F_\nu(v)$ and $G_\nu(v)$ in terms of series of modified Bessel functions of positive integral order are given by

$$F_\nu(v) = (\nu\pi/\sinh \nu\pi)^{\frac{1}{2}}[A(\nu, v)\cos\theta(\nu, v) + B(\nu, v)\sin\theta(\nu, v)],$$
$$G_\nu(v) = (\nu\pi/\sinh \nu\pi)^{\frac{1}{2}}[B(\nu, v)\cos\theta(\nu, v) - A(\nu, v)\sin\theta(\nu, v)],$$

where

$$\theta(\nu, v) = \nu \ln \tfrac{1}{2}v - \arg \Gamma(i\nu),$$

$$A(\nu, v) = \sum_{m=1}^{\infty} m(-1)^m (\tfrac{1}{2}v)^m I_m(v)/[m!(m^2 + \nu^2)],$$

$$B(\nu, v) = \sum_{m=0}^{\infty} \nu(-1)^m (\tfrac{1}{2}v)^m I_m(v)/[m!(m^2 + \nu^2)],$$

or

$$A(\nu, v) = -\sum_{n=1}^{\infty} \sum_{k=0}^{[\frac{1}{2}n - \frac{1}{2}]} \frac{(-1)^k(n)_{n-1-2k}\nu^{2k}v^{2n}}{4^n n!(1^2 + \nu^2)\cdots(n^2 + \nu^2)},$$

$$B(\nu, v) = \nu^{-1} + \nu^{-1}\sum_{n=1}^{\infty} \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k(n)_{n-2k}\nu^{2k}v^{2n}}{4^n n!(1^2 + \nu^2)\cdots(n^2 + \nu^2)},$$

where $[s]$ represents the greatest integer contained in s and the symbol $(p)_q$, where p and q are any positive integers such that $q \leqslant p$, denotes the sum of all the different products which can be formed by multiplying together q of the p factors $1, 2, \cdots, p$. $(p)_0$ being equal to 1 by definition. A short table of values of $(p)_q$ was given by M. BÔCHER, "On some applications of Bessel's functions with pure imaginary index," *Annals Math.*, v. 6, 1892, p. 144. [This table, $p = 1(1)8$, $q = 0(1)8$, was taken from O. X. SCHLÖMILCH, *Kompendium der*

höheren Analysis, v. 2, fourth ed., Brunswick, 1895, p. 31.] Definite integral representations of these functions are

$$F_\nu(v) = (\sinh \nu\pi)^{-1} \int_0^\pi e^{v \cos\theta} \cosh \nu\theta d\theta - \int_0^\infty e^{-v \cosh t} \sin \nu t dt,$$

(3) $$G_\nu(v) = \int_0^\infty e^{-v \cosh t} \cos \nu t dt.$$

Since the wedge functions have an oscillatory singularity at $v = 0$, it is more convenient to tabulate the related quantities $F_\nu(e^x)$ and $G_\nu(e^x)$ as functions of x. These latter functions satisfy the differential equation

(4) $$d^2w/dx^2 + (\nu^2 - e^{2x})w = 0,$$

obtained from (1) by the transformation of variable $v = e^x$, and $x = \ln v$, which takes the triad of points $(0, 1, \infty)$ of the v-axis into the triad $(-\infty, 0, \infty)$ of the x-axis. The functions $F_\nu(e^x)$ and $G_\nu(e^x)$ have no singularities on the finite part of the x-axis, and they approach sinusoids in νx as $x \to -\infty$ ($v \to +0$).

In the tables $\nu = .2(.2)10$, $x = -.49(.01)2.5$, to 5S mostly. $F_\nu(x)$ is tabulated over the complete ranges. In the region where $F_\nu(e^x)$ is oscillatory the error in the last figure given should not exceed 5 units. In the region where $F_\nu(e^x)$ is non-oscillatory the error in the tabulated values should not exceed 5 parts in 10 000.

The function $G_\nu(e^x)$ is tabulated over the following ranges in ν and x:

$$.2 \leqslant \nu \leqslant 1, -.49 \leqslant x \leqslant .5; \qquad 1.2 \leqslant \nu \leqslant 2, -.49 \leqslant x \leqslant 1;$$
$$2.2 \leqslant \nu \leqslant 4, -.49 \leqslant x \leqslant 1.5; \qquad 4.2 \leqslant \nu \leqslant 7, \quad .49 \leqslant x \leqslant 2;$$
$$7.2 \leqslant \nu \leqslant 10, \quad .49 \leqslant x \leqslant 2.5.$$

The error in the last figure of any tabulated value does not exceed 5 units.

As a matter of interest the value of $G_\nu(e^x)$ computed from the definite integral (3) and correct to the last printed figures, 5S, are given (leaf 61) for $x = 1(.5)2.5$, and for those values of ν not included in the main table.

                                        Extracts from the text

Bessel functions of imaginary order and imaginary argument provide solutions of Laplace's equation useful in certain potential and heat flow problems.[1,2] If it is desired to find a potential function depending on the cylindrical coordinates ρ, ϕ, z, which vanishes on the cylinders $\rho = \rho_1$, $\rho = \rho_2$, and on the planes $z = z_1$, $z = z_2$, and takes arbitrary values on the axial planes $\phi = \phi_1$, $\phi = \phi_2$, one assumes an infinite series of terms of the form

$$[A F_\nu(k\rho) + B G_\nu(k\rho)] \begin{matrix} \sinh \\ \cosh \end{matrix} \nu\phi \begin{matrix} \sin \\ \cos \end{matrix} kz,$$

and determines the separation constants ν and k and the coefficients A and B of each term to satisfy the given boundary conditions.

Some hydrodynamical investigations of the stability of flow of superposed streams of fluid in which density and velocity both vary with height have been carried out by TAYLOR[3] and GOLDSTEIN[4] in a form which leads to Bessel functions of imaginary order.

The propagation of transverse seismic waves over the surface of an elastically inhomogeneous medium is of considerable interest in geophysics, and it has been shown[5,6] that the transmission of these so-called LOVE waves is expressible in terms of Bessel functions of (large) imaginary order and imaginary argument if the modulus of rigidity of the elastic medium increases as a quadratic function of depth.

The flow of electric current between coaxial cylindrical electrodes, taking account of both convection and diffusion, has been investigated by BORGNIS[7] using the functions of purely imaginary order; and subsequently EMDE[8] has discussed in some detail asymptotic representations of Bessel functions whose order is large and imaginary.

Finally Bessel functions of imaginary order and imaginary argument occur in solutions of the wave equation, upon proper choice of the separation constants. These solutions arise

when one considers the propagation of acoustic or electromagnetic waves through bent pipes,[9] a problem wherein the functions of imaginary order and *real* argument also play an important role.

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[1] M. BÔCHER, *loc. cit.*
[2] J. DOUGALL, Edinb. Math. Soc., *Proc.*, v. 18, 1900, p. 33–83.
[3] G. I. TAYLOR, R. Soc. London, *Proc.*, v. 132A, 1931, p. 499–507.
[4] S. GOLDSTEIN, R. Soc. London, *Proc.*, v. 132A, 1931, p. 524–548.
[5] H. JEFFREYS, RAS, *Mo. Not., Geophys. Suppl.*, v. 2, 1928–31, p. 101–111.
[6] S. SAKURABA, Geophys. Mag., Tokyo, v. 9, 1935, p. 211–214.
[7] F. BORGNIS, *Ann. d. Physik*, s. 5, v. 31, 1938, p. 745–754.
[8] F. EMDE, *Z. f. angew. Math. u. Mech.*, v. 19, 1939, p. 101–118.
[9] P. KRASNUSHKIN, *Jn. Phys.*, S. S. S. R., v. 10, 1946, p. 443.

505[L].—NBSCL, *Table of* $J_0(2\sqrt{u})$, $Y_0(2\sqrt{u})$, $u^{-\frac{1}{2}}J_1(2\sqrt{u})$, $u^{-\frac{1}{2}}Y_1(2\sqrt{u})$. Nov. 1947, ii, 11 leaves. 20.6 × 35.6 cm. Hectographed preliminary typescript on one side of each leaf, not available for general distribution.

These tables of $J_0(2u^{\frac{1}{2}})$, $Y_0(2u^{\frac{1}{2}})$, $u^{-\frac{1}{2}}J_1(2u^{\frac{1}{2}})$, $u^{-\frac{1}{2}}Y_1(2u^{\frac{1}{2}})$ are for $u = [0(.05)3(.1)5(.2)$-$8(.5)50(2)160(5)410; 8–11D]$. For most of the values tabulated last-place accuracy is not guaranteed. Second, or second and fourth, differences are given everywhere except for $u > 160$, where the interval in u is too large for interpolation, and near the origin, where $Y_0(2u^{\frac{1}{2}})$ and $u^{-\frac{1}{2}}Y_1(2u^{\frac{1}{2}})$ have singularities. In the regions where differences are not given it is best to use, for interpolation purposes, the tables of $J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$, given in BAASMTC, *Math Tables*, v. 6 or, for $2u^{\frac{1}{2}} < 1$, the NBSCL tables of $Y_0(x)$ and $Y_1(x)$ at interval .0001 (in the press).

Tables of $J_0(2x^{\frac{1}{2}})$ and $x^{-\frac{1}{2}}J_1(2x^{\frac{1}{2}})$, [by J. R. AIREY], were published in BAAS, *Report 1924*, p. 287–295, for $x = [0(.02)20; 6D]$.

<div align="right">*Extracts from text*</div>

506[L].—U. S. NAVY, OFFICE OF RESEARCH AND INVENTIONS, *Scattering and Radiation from Circular Cylinders and Spheres; Tables of Amplitudes and Phase Angles*, prepared by A. N. LOWAN for NBSCL, and P. M. MORSE, H. FESHBACH, and M. LAX for the M.I.T. Underwater Sound Laboratory. New edition, July, 1946, vi, 124 p. 19.8 × 25.9 cm. See *MTAC*, v. 1, 1945, p. 390.

In its endeavor to disseminate scientific information the Office of Research and Inventions has published this Report, formerly restricted in availability, in a compact new edition (smaller paper and type page, 124 p. instead of 124 leaves), which ought to be of considerable value to physicists and engineers engaged in acoustical and electromagnetic wave research. We refer above to our review of the original edition. The reprint is so exact that the erroneous title-entries for tables 10 and 11 in the Table of Contents, which we earlier noted, still persist.

507[L, M, I].—HERBERT E. SALZER, "Table of coefficients for repeated integration with differences," *Phil. Mag.*, s. 7, v. 38, May 1947 [publ. Oct. 1947], p. 331–338.

The exact values of $G_n^{(1)} = (1/n!)\int_0^t \int_0^t t(t - 1) \cdots (t - n + 1)(dt)^2$ and $H_n^{(1)} = (1/n!)$ $\times \int_0^t \int_0^t t(t + 1) \cdots (t + n - 1)(dt)^2$ are given for $n = 1(1)20$. For the same values of $n \leqslant 22 - k$ are also given $G_n^{(k)} = \int_0^t \cdots \int_0^t \int_0^t t(t - 1) \cdots (t - n + 1)(dt)^k/n!$, for $k = 2$, 12D; 3, 10 or 11D; 4, 9–11D; 5, 8–11D; 6, 7–10D. Also $H_n^{(k)} = (1/n!)\int_0^t \cdots \int_0^t \int_0^t t(t + 1)$ $\cdots (t + n - 1)(dt)^k$, $k = 2$, 11D; 3, 9–10D; 4, 8–10D; 5, 7–10D; 6, 6–9D.

A table of $G_n^{(1)}$ and $H_n^{(1)}$, $n = 1(1)20$, was given by A. N. LOWAN & H. E. SALZER,[1] *Jn. Math. Phys.*, v. 22, 1943, p. 49–50. $G_n^{(2)}$ and $H_n^{(2)}$ for $n = 1(1)7$ were given by W. E. MILNE, "On the numerical integration of certain differential equations of the second order," *Amer. Math. Mo.*, v. 40, 1933, p. 324.

Extracts from text

[1] Compare *MTAC*, v. 1, p. 157.—EDITOR.

508[M].—A. A. DORODNÍTSYN, "Asimptoticheskoe reshenie uravneniía Van-der-Polía" [Asymptotic solution of Van der Pol's equation]. *Prikladnaía Matem. i Mekhanika. Applied Math. and Mechanics*, v. 11, July 1947, p. 313–328, table p. 327. 16.8 × 25.8 cm.

The table gives 4D values of $Q_0(u)$ and $Q_1(u)$ for $u = -6(1)-4(.2)+4$.
$Q_1(u) = (1/A(u))\int_u^\infty A(u)(uQ_0^{-1} - u^2)du$, $A(u) = \exp(-\int_0^u du/Q_0^2)$,
$Q_0(u) = du/d\tau = u^2 - \tau = u^2 + \tau_1$, where
$u = \sqrt{\tau_1}\{J_{-3/2}(\tfrac{2}{3}\tau_1^3) - J_{3/2}(\tfrac{2}{3}\tau_1^3)\}/\{J_{1/3}(\tfrac{2}{3}\tau_1^3) + J_{-1/3}(\tfrac{2}{3}\tau_1^3)\}$
is a solution of the equation $d^2u/d\tau^2 - 2udu/d\tau + 1 = 0$.

509[M].—G. PLACZEK, "The angular distribution of neutrons emerging from a plane surface," *Phys. Rev.*, s. 2, v. 72, Oct. 1, 1947, p. 556–558. 20 × 26.7 cm.

Of $\phi(u) = \tfrac{1}{2}(1 + u)^{-\tfrac{1}{2}}e^w$, $w = \pi^{-1}\int_0^u x \tan^{-1}(u \tan x)dx/(1 - x \cot x)$, there are two tables, by NBSCL: T.I, $u = [0(.1)1 ; 7D]$, obtained by numerical integration; and T.II, obtained from these values by interpolation $u = [0(.01)1 ; 5D]$, Δ, and Δ^2, 0 to .19. u is the cosine of the angle between the direction of the motion of the neutron and the outward normal.

510[M].—Miss P. M. SKINNER (Mrs. TRUSCOTT), "Numerical tables," in A. HAMMAD, "The primary and secondary scattering of sunlight in a plane-stratified atmosphere of uniform composition.—Part III. Numerical tables and discussion of secondary scattered light," *Phil. Mag.*, s. 7, v. 38, July 1947 (publ. Nov. 1947), p. 515–529. 17 × 25.1 cm.

T.I, p. 518–519, $Ej_n(x)$, for $n = 1(1)5$, $x = [0(.01)1 ; 6D]$; $Ej_n(x) = \int_1^\infty e^{-xt}t^{-n}dt$; $Ej_1(x) = -Ei(-x)$. The values for $n = 2(1)5$ were obtained from the values for $n = 1$ by applications of the recurrence formula $(n - 1)Ej_n(x) = e^{-x} - xEj_{n-1}(x)$, $n > 1$.

T.II, p. 520–524, $L_n(c, z, \overline{m})$, for $c = .1, .2(.2)1$, $n = 1, 3, 5$,
$\overline{m} = 1 - m = 0(.2)1$, sec $z = [1(1)4, 1.5, 6 ; 5D]$;
$L_n(c, z, m) = c^{-1}e^{z-1}[Ek_n(cm, -\sec z) + Ek_n(c\overline{1 - m}, \sec z)]$, where
$Ek_n(r, b) = \int_0^r e^{bt}Ej_n(t)dt$, $Z = c \sec z$.

T.III, p. 527, $e^\Theta Ld_n(c, z, \overline{m} = 1, \theta)$, $n = 1(2)5$, sec $z = 1, 2, 4$,
sec $\theta = [1, 2(2)6, \infty ; 5D]$, $c = .2(.2)1$, $\Theta = c \sec \theta$;
$Ld_n(c, z, \overline{m}, \theta) = e^{-\Theta \overline{m}}\int_0^{\overline{m}}e^{-\Theta m'}L_n(c, z, m')d\overline{m}'$.

Extracts from text

EDITORIAL NOTE: A table of $Ej_1(x)$ was given by J. W. L. GLAISHER, R. Soc. London, *Trans.*, v. 160, p. 371, 380–381, 385–386, $x = [0(.01)1 ; 18D]$, Δ^2, $[1(.1)5 ; 11D]$, Δ^3, $[6(1)15 ; 11D]$, $[2 ; 43D]$, $[20 ; 12D]$. Tables of $Ej_n(x)$ were given by E. GOLD, R. Soc. London, *Proc.*, v. 82A, 1909, p. 62, for $n = 1(1)3$, $x = [.01(.01).1(.05)1(.1)2(.2)3(.5)5,-6 ; 5D]$. Hammad's description of this range is very incorrect and Glaisher's table, p. 380–381, would never be found from the reference he gives for it. See also FMR, *Index*, p. 207. Miss Skinner's values for $Ej_1(x)$ agree with rounded-off values of Glaisher, except that Glaisher's $Ej_1(.46)$ would produce 0.611387 rather than Miss Skinner's 0.611386. The 9-place table of NBSCL shows Glaisher to be the more accurate.

511[M].—O. A. Tsukhanova & G. D. Salamandra, "Rasprostranenie tepla ot sfericheskogo istochnika, okhlazhdaemogo v sypuchei srede," [Diffusion of heat from a spherical source, cooled in a granular medium]. Akad. N., SSSR, Leningrad, *Izvestiia, Otd. Tekhn. N.*, no. 8, Aug. 1947, p. 977–986, tables, p. 983. 16.5 × 25.8 cm.

There are six tables of $J = \int_c^\infty e^{\alpha} d\alpha$, where $y = (\beta/\alpha)^2 - \alpha^2$, for (i). $\beta^2 = .001$, $c = [.01, .02, .06, .1(.1)1.5; 6D]$; (ii). $\beta^2 = .01093$, $c = [.056, .066(.02).106(.04).386, .466, .586, .746, .946, 1.186, 1.427;$ mostly 4D]; (iii). $\beta^2 = .04$, $c = [.1(.06).4(.1)2.5; 6D]$; (iv). $\beta^2 = .1$, $c = [.18(.04).5(.1).8; 6D]$, $[.1(.1)2.8; 7D]$; (v). $\beta^2 = .5$, $c = [.42(.06).6(.1)1.5; 6D]$, $[1.6(.1)2.5; 7D]$; (vi). $\beta^2 = 2.4806$, $c = [.87(.1)1.57; 4–5D]$.

512[M, V].—S. A. Khristianovich, "Priblizhennoe integrirovanie uravnenii sverkhzvukovogo techeniia gaza" [Approximate integration of equations of a supersonic gas flow], *Prikladnaia Matem. i Mekhanika. Applied Math. and Mechanics*, v. 11, June 1947, p. 215–222; tables p. 217. 16.8 × 25.8 cm.

There are tables (mostly 4S) of $\sqrt{x} = \sqrt{\{(\lambda^2 - 1)/[1 - \lambda^2(\kappa - 1)/(\kappa + 1)]^y\}}$, $y = (\kappa + 1)/(\kappa - 1)$, and $\sigma' = \int_1^\lambda \{(t^2 - 1)/[1 - t^2(\kappa - 1)/(\kappa + 1)]\}^{\frac{1}{2}} dt/t$, for $\lambda = 1(.05)$-$1.5(.1)2.4$; $\kappa = 1.405$.

513[Q].—Deutsche Seewarte, *Ortsbestimmung durch astronomische Beobachtung zweier gegebener Fixsterne mit Hilfe von Höhengleichendiagrammen*, prepared for and issued by the Oberkommando der Kriegsmarine, 1940–1941. Loose leaf, 8 v., 171 + 161 + 165 + 189 + 150 + 146 + 170 + 147 p. 21 × 30 cm.

These volumes were issued at inervals from January 1940 to September 1941; reference is made to volumes for southern latitudes but none is known to have been issued.

The principal use of star altitude curves is to determine position, more particularly latitude and local sidereal time, from the known altitudes of two (or more) stars. Accordingly curves of constant altitude, for two or more stars, are superposed upon a rectangular framework of latitude and local sidereal time, from which both can be read corresponding to any point defined by the altitudes of two or more stars. The classic publication in this field is Weems' *Star Altitude Curves* (see *MTAC*, v. 2, p. 133–134), and comparison is inevitable.

In the volumes under review, the latitude scale (arranged vertically) is linear, 2 cm., to each degree; the horizontal sidereal time scale is 0.5 cm. to a minute of time, half-an-hour to a page. The scale varies slightly from volume to volume and is rather larger in the later (lower latitude) volumes. A rectangular grid of pecked lines (in the early volumes, continuous) is drawn for every 10′ of latitude and $0^m.5$ of sidereal time; the lines for every degree and multiple of 5^m are emphasized. Upon this background are overprinted two sets of star altitude curves, given for every 10′, with the integral and half-integral degree curves emphasized. The border scales are adequately indicated and divided, but the altitude curves are only marked (once) for each degree and this seems insufficient; the pages are well supplied with catch headings. The scale used enables positions to be read to within about 1′ or 2′ in latitude and about $0^m.1$ or $1′.5$ in longitude.

In any one volume, any number from six to eighteen different star pairs may be used; an index table (or rather series of diagrams) in the front of each volume indicates the coverage and the pairs that are available at any time. At least two, but usually more, star pairs are always available. The volume is tab-indexed according to star-pair, which are arranged in (overlapping) order of coverage. In some of the later volumes the altitude of *Polaris* is indicated by scales alongside the latitude scales on each side of the page.

The various auxiliary tables and diagrams include a star map, corrections for annual variation in altitude due to precession (the positions of the stars used are those of 1941.0), corrections for refraction at heights above the Earth's surface—the curves as drawn include mean refraction at sea level—tables of sidereal time at 0^h U.T. for various years, various tables for transferring position lines, and finally diagrams for determining approximate azimuths of the stars used.

In Weems' diagrams the latitude scale is not constant, but varies as a Mercator chart. This means that non-simultaneous position lines can be transferred in the normal manner. In these diagrams a constant latitude scale is adopted, and the rules for transferring position lines are necessarily more complicated. Tables 8, 9 and 10 provide the means whereby this is accomplished. For a two star fix, a point on the diagram is obtained as if the two sights were simultaneous. This point is now transferred horizontally along the time scale by an amount equal to the time difference between the sights. From this position a line is drawn whose azimuth depends on the latitude and the course of the aircraft (Table 9). A point on this line is now chosen whose distance from the transferred point depends on time difference and speed of aircraft (Tables 8 and 10). The final fix is obtained by moving this point parallel to the curves corresponding to the first star sighted until the altitude of the second star is consonant with the observed value.

Five pages of explanation and examples are given covering all problems likely to arise in astronomical air navigation.

The early volumes are comparatively crude compared to the later ones and it is clear that many technical advances were made in the course of the production. The later volumes are on a slightly larger scale, printed on better paper and with thinner and clearer lines; although both background and superposed grids are printed in black, there is surprisingly little confusion and there is no doubt that the background grid aids reading of the scales. Weems has no grid and his star curves are beautifully printed in contrasting colours; the *Polaris* curves are liable to mislead in this respect since at first glance they appear to be lines of constant latitude. A combination of the two would seem to provide the optimum method of presentation.

There is a mystery about the production of these volumes which underlines the comparison with Weems' star curves; why should the German OKM produce eight (at least) large unwieldy volumes, clearly designed primarily for air use? There is no indication that they were intended for or used by the Luftwaffe, which was amply provided for by other tables. The difference in bulk, and in elegance, between these volumes and those of Weems is very marked; some of this disparity is accounted for by the fact that, on the average, three star pairs are catered for throughout as against Weems' one pair and *Polaris* (for northern latitudes), and the remainder by the thick paper and the awkward loose leaf binding.

This is a case where graphical or diagrammatical presentation has a distinct advantage over the tabular methods. Direct tabulation of latitude and local sidereal time with the two altitudes as argument is unsatisfactory; it covers two awkward-shaped areas on the Earth's surface and involves double-entry interpolation which is only linear if a small interval of tabulation is used. Tabulation of the two altitudes with arguments latitude and local sidereal time would involve double inverse interpolation, which, even if linear, cannot be attractive.

<div style="text-align:right">J. B. PARKER & D. H. SADLER</div>

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Bath, England

514[Q].—JAPAN, HYDROGRAPHIC DEPARTMENT, *Altitude and Azimuth Almanac for August–December 1947* (Title also in Japanese). Tokyo, May, 1947, vi, 93 p. + 1 plate. 18.4 × 26 cm.

This almanac is unique among published ephemerides in that it tabulates directly the altitude and azimuth as would be observed from certain specified positions on the surface

of the Earth. It appears to be the successor to a war-time publication "Publication No. 9582, Altitude-Azimuth Tables for September 1945 (Sun, Moon, Planets and Fixed Stars) computed for Kisarazu, Naha, Iwo Jima and Kanoya and for Japanese Central Standard Time";[1] however, whereas the earlier publication was designed for air navigation and catered for all celestial bodies, the present almanac is designed for surface navigation and is restricted to the Sun.

Briefly, the altitude, to the nearest minute, and azimuth, to the nearest degree, of the Sun's lower limb are tabulated for every 10 minutes (occasionally 20 minutes) of Japanese Central Standard Time (Time Zone -9^h) for longitude E. 150° and four latitudes, N. 25°, N. 30°, N. 35° and N.40°; the altitude is corrected by application of semi-diameter, parallax, refraction and dip from a height of 5 meters, so as to be directly comparable with the observed altitude, corrected only for instrumental errors. Tabulations are given whenever the Sun is above the horizon, but the interval of tabulation is increased to 20^m earlier than 6^h and later than 17^h in order to get all the tabulation for one day in one column, containing a maximum of 72 or 73 lines arranged in blocks of 6 (occasionally the last block for the first days in each section contains 7 lines). There are eight such columns on each page, containing the tabulations for eight days for one latitude; each latitude constitutes a separate section.

In addition, there is a one-page table for each latitude giving the altitude of *Polaris* in longitude E. 150°, for every 20^m of standard mean time during darkness and for every sixth day; the altitude is corrected for refraction and dip. There is also a short table of the additional dip correction to be applied if the observation is made from a height differing from 5 meters, and a one-page interpolation table giving ten sub-multiples of the 10^m differences in the range $0°(2')5°$. The range of the latter table is surprising as the maximum difference for a 10^m interval is $2°30'$.

The preface, explanation, page headings and in fact everything but the actual figures are in Japanese. The instructions and examples are not difficult to follow, though in one case the reviewer has not been able to reproduce the figures given. The general method of use advocated is to adopt as an assumed position the nearest point with both latitude and longitude multiples of 5°. Interpolation for longitudes other than E. 150° is done by applying an appropriate correction (multiple of 20^m) to the time; some examples seem to advocate correcting the observed altitude graphically for the change in time necessary to allow direct comparison with a tabular entry. In all cases long intercepts may occur.

The Almanac is not very well printed and the figures are rather small for easy reading; the chief fault is, however, a certain unevenness in alignment. Casual examination has failed to find any large errors, though one or two had already been corrected by pen and the end-figure does not always appear too reliable.

The advantages of being able to obtain an intercept by a straight comparison of an observed and a tabulated altitude are so great that it is well worth while considering carefully this form of almanac. As presented here it is intended to cover the relatively small area round Japan, but any universal almanac must cater for the whole of the Earth. Extension to different longitudes is not difficult, since this could be accomplished by using L.M.T. as argument and interpolating for longitude between consecutive days, as is done now for times of rising and setting; this correction cannot strictly be ignored even for the Sun with small longitude differences of the order of 15°, but for the Moon it would be very large and not linear.

Interpolation for latitude, to avoid long intercepts and if necessary to use the D.R. position, can be done fairly quickly and accurately from a knowledge of the azimuth, which is tabulated; but tabulations would have to be given for every 5° of latitude and this would seem to make the method impracticable for a universal almanac. Furthermore an interval of 10^m in time gives rise to quite large second differences in the altitude, so that linear interpolation (as is pointed out in one of the examples) may lead to considerable error; the answer would be a still smaller interval—say of 5^m—near the meridian.

Taking as a basis of comparison an air almanac tabulating G.H.A. and Dec., at intervals of 10^m, for the Sun, Moon and planets (the method is not suitable for the stars) it is inter-

esting to find the amount of tabulation required for this method for one latitude. Making reasonable allowances for the known periods of visibility to which tabulations can be restricted, to the small interval required near meridian passage and to the necessity for tabulating the Moon at intermediate longitudes, it works out about one-half the equivalent air almanac. This assumes a tabular accuracy of 1'; an increase to 0'.1 would make interpolation too difficult at the intervals of tabulation suggested. It is thus seen that an almanac of this nature would be an expensive luxury even for a comparatively narrow band of latitude.[2]

D. H. SADLER

[1] See *MTAC*, v. 2, p. 83, RMT 297, by Professor C. H. SMILEY.
[2] Reprinted, by permission, from Institute of Navigation [British], *Jn.*, v. 1, no. 2, 1948.

515[Q].—ZDENĚK KOPAL, *Theory and Tables of the Associated Alpha-Functions.* (Harvard College Observatory, *Circular* 450), Cambridge, Mass., 1947, 25 p. 24.6 × 29 cm.

The study of variable stars has always been of great interest to the astronomer. A variable star is one whose light intensity or magnitude varies with time. These variations may be self-repeating in a periodic manner as in the case of either the Cepheid-type variables or eclipsing variables, or the light variations may be semi-regular or irregular as in the case of the novae or supernovae. Conclusions of the greatest importance have been drawn from the study of variable stars despite little or no understanding of the details of the mechanism causing the light variation. For example, it was empirically discovered that there was a correlation—the so-called period-luminosity law—between the period of a Cepheid variable and its luminosity, or absolute magnitude. This "law" was used first, to determine the size of our own galaxy or Milky Way system and second, to determine the distances to some of the nearer of the external galaxies; and yet no satisfactory quantitative explanation of the details of the light-curve of a Cepheid has so far been made.

The theory and tables under review are primarily concerned with the theoretical interpretation of the light-curves of *eclipsing* variables. These eclipsing variables, or eclipsing binaries as they are sometimes called, are the only variable stars whose light-curves are subject to precise analysis and interpretation and, strangely enough, eclipsing variables are stars that are not *inherently* variable. Even in our largest telescopes an eclipsing binary looks like a single star; but each such star is actually a pair of stars revolving about a common center of mass, and through the accident of geometry mutually eclipsing one another every half revolution. Stars showing a periodic variation of their radial (line of sight) velocity are known as spectroscopic binaries and every eclipsing binary is also a spectroscopic binary, the light and velocity varying in the same period and in a manner consistent with the hypothesis of a binary system. The interpretation of the combination of photometric and spectroscopic observations of eclipsing binaries has yielded astronomical data of the greatest importance in connection with the determination of the diameters, volumes, masses, densities, temperatures, luminosities, rotations and internal constitutions of the stars.

The original methods of solution of a light-curve of an eclipsing variable were based on the simplest possible assumptions, namely, spherical stars presenting uniformly bright discs and moving in circular orbits. It was discovered that tables easily could be constructed giving α, the fractional loss of light (proportional to the eclipsed area in this simple case) as a function of k, the ratio of the radii of the two stars ($k \leqq 1$), and p, the geometric depth of eclipse, p being equal to $+1$ at first or external contact and equal to -1 at second or internal contact. From these tables it was possible to construct a theoretical light-curve corresponding to any combination of assumed values of r_1 (radius of the star undergoing eclipse), r_2 (radius of the eclipsing star), i, the inclination of the orbit plane, and P the period, r_1 and r_2 being taken in terms of the radius of the orbit as unity. The astronomer's

problem was to pick out from this infinity of theoretical light-curves that particular one which best satisfied his observations. Certain auxiliary functions of p and k were tabulated which greatly assisted this process of selection and made the solution of a light-curve a task of but a few hours.

Observations of the sun revealed that the surface brightness at a point near the edge of the disk, or limb, was considerably less than that at the center of the disk, and that the variation of brightness over the disk followed quite accurately the so-called cosine law of limb darkening, the amount of darkening at the limb being a function of the wave-length (color) of observation.

Tables for eclipsing variables, accurate to 3S, were computed on the hypothesis of complete darkening at the limb by RUSSELL & SHAPLEY[1] in 1912 by graphical integration, the analytical solution being one of great complexity involving complete and incomplete elliptic integrals of the first and second kind. It was discovered that either the uniform or darkened hypothesis would satisfy equally well almost all of the observed light-curves then available. This was essentially due to the poor photometric quality of even the best photographic or visual photometric observations, the percentage errors of single observations being about five or ten percent, whereas, in positional work for example, the errors may be perhaps as low as one part in five million.

A number of complications arise whenever the two components in an eclipsing system are very close together. The stars are distorted from spherical figures into approximately tri-axial ellipsoids by the strong tidal and rotational forces involved. The surface brightness of the disk at any point is not only a function of the distance from the center (or limb) but is also a function of local gravity at the point in question, the disk being darker at the equator than at the poles. Furthermore, the hemispheres that face each other will be brighter (by mutual absorption and re-radiation of the other component's radiant energy) than the opposite hemispheres. This last complication, the so-called "reflection" effect, has as yet been analyzed only under the most simplified assumptions and is a problem for the future. Dr. Kopal's paper deals with the effects on the light-curve—and also the radial velocity curve— "of the rotational and tidal harmonic distortions of components in close eclipsing systems— between minima as well as within eclipses—taking account of the appropriate distribution of brightness over the distorted surfaces (due to limb- and gravity-darkening)." This work has been stimulated by recent improvements and developments in the stability and sensitivity of the photoelectric cell and the multiplier phototube that make it possible to observe a large percentage of the two thousand known eclipsing variables with an accuracy of an entirely new order of magnitude.

If we let r_1 and r_2 denote the eclipsed and eclipsing radii of two spherical components of a binary system, and δ the instantaneous apparent separation of their centers projected on the celestial sphere, then the fractional loss of light can be conveniently expressed in terms of a family of associated alpha-functions α_n^m of various integral orders m and indices n which are non-dimensional quantities defined as

$$\pi r_1{}^{m+n+2}\alpha_n^m = \left\{ \int_s^{r_1} \int_{-[r_1{}^2-x^2]^{\frac{1}{2}}}^{[r_1{}^2-x^2]^{\frac{1}{2}}} + \int_s^s \int_{-[r_2{}^2-(\delta-x)^2]^{\frac{1}{2}}}^{[r_2{}^2-(\delta-x)^2]^{\frac{1}{2}}} \right\} x^m z^n dx dy$$

if the eclipse is partial, and

$$\pi r_1{}^{m+n+2}\alpha_n^m = \int_{\delta-r_2}^{\delta+r_1} \int_{-[r_2{}^2-(\delta-x)^2]^{\frac{1}{2}}}^{[r_2{}^2-(\delta-x)^2]^{\frac{1}{2}}} x^m z^n dx dy$$

if it is annular, where

$$z = (r_1{}^2 - x^2 - y^2)^{\frac{1}{2}} \qquad \text{and} \qquad s = (r_1{}^2 - r_2{}^2 + \delta^2)/2\delta.$$

An inspection of these equations reveals that all associated alpha-functions may be made to depend upon two non-dimensional variables (such as k and p) and that 91 bivariate

tables would be necessary for the second, third and fourth harmonic distortions of the photometric and radial velocity curves. The author introduces a new integral which he calls an I-integral defined by

$$\pi r_2{}^q I_{\beta,\gamma}^m = 2^{\frac12\gamma} \int_{\delta-r_2}^c [r_2{}^2 - (\delta - x)^2]^{\frac12\beta}(s - x)^{\frac12\gamma}(\delta - x)^m dx$$

where $c = s$ or $\delta + r_2$ for partial or annular eclipse respectively, and $q = 1 + \beta + \frac12\gamma + m$. The integral $I_{\beta,\gamma}^m$ is a non-dimensional quantity depending on a *single* variable and all associated alpha-functions of orders $m = 0(1)3$ can be expressed as simple combinations of the I-integrals factored by powers of r_2/r_1 and δ/r_2 or s/r_2.

Dr. Kopal gives the relationships and recurrence formulae necessary to transfer from the alpha-functions to the I-integrals. The properties of I-integrals are then discussed and their evaluation in terms of the complete elliptic integrals of the first and second kind and the variable $\mu = (\delta - s)/r_2$ is next given. It is shown that the use of μ as an argument in the tables would be unwise and a new variable α is introduced, defined by

$\mu = 1 - 2 \sin^2 \frac12\alpha$ for partial eclipse, and $\mu = 1 - 2 \csc^2 \frac12\alpha$ for annular eclipse.

The introduction of α as argument and the use of modified second differences defined by

$$M'' = \Delta'' - 0.184\Delta^{iv} + \cdots$$

permits the tabulation of a tremendous amount of information in a relatively small space. The author then gives a numerical example concerning a fictitious eclipsing system. Diagrams are next given, which illustrate the behavior of various I-integrals during partial eclipse. Finally the following I-integrals are tabulated to 5S (in general): $I_{-1,0}^m$ and $I_{-1,1}^m$ for $m = 0(1)5$; $I_{-1,\gamma}^m$ for $\gamma = 2(1)7$; $I_{1,\gamma}^0$ for $\gamma = -1(1) + 6$; and $I_{3,\gamma}^0$ for $\gamma = -1(1) + 3$. These integrals with their modified second differences are tabulated for both partial and annular eclipses (except for $I_{-1,0}^m$ which is tabulated for partial eclipse only) with $\alpha = 0(5°)180°$. Many of these integrals become infinite for $\alpha = 0°$ and interpolation for small α becomes difficult or impractical. The reviewer feels that the tabulation of the reciprocals of the I-integrals, or perhaps the use of a factor such as δ, might be a considerable improvement for these portions of the tables.

This is a very timely and important piece of work. Although the scope and applicability of these new functions are still far from being fully explored, it appears certain that little real progress in the interpretation of a wide range of phenomena observed in close binary systems can be expected until the general properties of the associated alpha-functions are well understood. The author indicates that the functions will be used to predict the light or velocity changes of configurations in free non-radial oscillations, as well as to interpret the forms of line profiles of rotating distorted stars in full light or during partial or annular eclipse. The problems encountered in such close systems are formidable both from the point of view of the astrophysicist and of the mathematician. Dr. Kopal is to be congratulated upon making such a long step forward in the solution of such problems.

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[1] H. N. RUSSELL & H. SHAPLEY, "On darkening at the limb in eclipsing variables," *Astrophysical Jn.*, v. 36, 1912, p. 239–254, 385–408.

516[Q].—O. A. DE AZEREDO RODRIGUES, *Tábuas para Retas de Altura* (Escola Naval no. 23). Rio de Janeiro, Imprensa Naval, 1943, 54 p. 16 × 23.6 cm. Approved for printing, 3 May 1940, by the Minister of Marine.

This volume is quite similar to FONTOURA DA COSTA and PENTEADO's *Tábuas de Altura e Azimute*[1] (later referred to as F & P) and to AGETON's *Dead Reckoning Altitude and*

Azimuth Table (H.O. 211, RMT 104), both in contents and in arrangement. The astronomical triangle is divided into two right triangles by a perpendicular from the celestial body upon the meridian. As in F & P, the interval of the argument is one minute of arc, and it is the values of $C = 10^5 \log \csc x$ which are printed in heavy type rather than those of $S = 10^5 \log \sec x$ as in Ageton. This will tend to confuse persons already well acquainted with the latter volume. Values of S and C less than 665 are given to one decimal; in Ageton, only values less than 240 are given to one decimal.

The formulae are essentially the same as in the two earlier volumes; only a slight change in notation is involved. The author calls the length of the perpendicular from the celestial body upon the meridian G (Ageton's R, F & P's $90° - \psi$) rather than $90° - G$; the declination of the foot of the perpendicular, $90° - V$ (Ageton's K, F & P's $90° - \gamma$). As in F & P, the author neglects to warn the user of the inaccuracies which may arise when the foot of the perpendicular lies near the pole.

The arrangement of the tables is almost identical with that of F & P. The only differences noted were that the C column is printed to the left of the S column; the arguments given at the foot of the page are only those from 90° to 180° (the ones from 270° to 360° given in parentheses in F & P are omitted); there is no convenient thumb-tab index.

The usual tables for refraction, dip of the horizon, and parallax are presented early in the volume. Also included are a table of the six natural trigonometric functions $[0(1°)90°; 4D]$, a log sine table $[0(10')90°; 4D]$, a log tangent table $[0(10')89°50'; 4D]$ and 4-place tables of logarithms of numbers and anti-logarithms.

One uncommon useful feature of this volume is a nomogram to be used in correcting a · circum-meridian altitude. The last sixteen pages of the volume are devoted to explanations of the use of the tables and nomogram.

A comparison of 1200 values of C and S in this book with the corresponding ones in F & P revealed no differences.

<div align="right">CHARLES H. SMILEY</div>

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¹ See, in this issue, N 92.

517[U].—THOMAS F. HICKERSON. *Latitude, Longitude and Azimuth by the Sun or Stars.* Chapel Hill, N. C., published by the author, 1947, ii, 101 p. 13.5 × 23.4 cm. $2.00.

This volume is a second edition of *Navigational Handbook with Tables*, published by the author at Chapel Hill in 1944. It differs from the earlier volume in that it contains 50 pages of explanation instead of 29, and includes a table (I) of meridional parts to 0.1 with argument latitude, $0(10')79°$, $79°(2')79°58'$ condensed from table 5 of BOWDITCH, 1938 edition, which is identical with table 3 of Bowditch, editions of 1903 up to that of 1938. There are also tables for Interval to Noon with "Ship moving Westward" (Table III) and "Ship moving Eastward" (Table IV).

The principal table (II) is similar to that given in Ageton's Dead Reckoning Altitude and Azimuth Table, with values of $A = 10^5 \log \csc x$ and $B = 10^5 \log |\sec x|$ generally given to the nearest integer, but with one decimal given for all tabular values less than 106. There are two major exceptions to this similarity: the interval of the argument is $0'.2$ instead of $0'.5$ and the table does not follow Ageton's pattern in which the letters at the top and bottom of a column are the same and each tabular value appears twice. This table folds back on itself at 45° instead of at 90°. Other tables included are: Conversion of Arc into Time (V), Time into Arc (VI) and Altitude Corrections (A–C). The latter table appears to have been taken directly from the *American Nautical Almanac*.

In the explanation of the use of the tables, the material is presented as a number of "cases": Case 1, Great Circle Course and Distance; Case 2, Points along a Great Circle Track; Case 3, Line of Position with Appendix giving the "Fix," etc.

The work forms given are essentially those in Ageton, plus about eight new ones. No warning is given of the difficulties which arise when the foot of the perpendicular lies near a pole; the smaller interval of the argument makes this less serious than in Ageton's tables.

The author's Case 7 (Latitude when the Sun or Star is near the Meridian) and Case 9 (Longitude when the Sun or Star is near the Prime Vertical) will probably appeal to many practical navigators. It is unfortunate that the author did not warn the reader of the troubles which arise in these cases if the altitude is near 90°.

The formula used in Case 9 is

$$2B(t/2) = B(s) + B(s - z) - B(d) - B(L)$$

where t and d are the meridian angle and declination respectively of the celestial body, L is the observer's latitude, z is $90° - H$ or the body's zenith distance, $s = (z + d + L)/2$. Table S, based upon this formula, gives t and z for Betelgeuse, each to the nearest minute of arc for L, $0(1°)60°$ and H, $20°(5°)35°$. This table is quite similar to, and possesses many of the advantages of, *Tafeln zur astronomischen Ortsbestimmung*, by ARNOLD KOHLSCHÜTTER, Berlin, 1913. A set of similar star tables prepared today for the northern temperate zone would make a valuable addition to navigational literature. They would possess the great advantage of a direct approach, allowing the navigator to enter a table with the observed altitude as an argument, and with an assumed latitude or longitude, find the corresponding longitude or latitude.

A brief examination of the principal table (II) indicates that the tabular values are much more reliable than those in Ageton.

<div align="right">CHARLES H. SMILEY</div>

518[V].—G. MORETTI, "Scie piane turbolente," *L'Aerotecnica*, v. 27, 15 June 1947, p. 210–221. 20.5 × 29 cm.

The tables, p. 219–220, are (1) of $M(x|\alpha, \xi)$, $N(x|\alpha, \xi)$, $P(x|\alpha, \xi)$, $\alpha = 1.069$, $\xi = .765$, for $x = [0(.1)2, 2.2, 2.5, 3; 3–4D]$, where

$$M = [\alpha^2 e^{-x^2}/C(\xi)] \begin{vmatrix} A(x)C(x) \\ A(\xi)C(\xi) \end{vmatrix}, \qquad N = [\alpha^2 e^{-x^2}/C(\xi)] \begin{vmatrix} B(x)C(x) \\ B(\xi)C(\xi) \end{vmatrix},$$

$$P = [2\alpha e^{-x^2}/C(\xi)] \begin{vmatrix} x^2 - \frac{1}{2} & C(x) \\ \xi^2 - \frac{1}{2} & C(\xi) \end{vmatrix}, \qquad A(x) = \frac{1}{2}e^{-x^2} + x \int_0^x e^{-t^2}dt,$$

$$B(x) = \frac{1}{2}e^{-x^2} - x \int_0^\infty e^{-t^2}dt, \qquad C(x) = \frac{1}{2}e^{x^2} - x \int_0^\infty e^{t^2}dt;$$

(2) of $y(x|\alpha, \xi, \mu) = M(x|\alpha, \xi) + \mu N(x|\alpha, \xi)$ and of $y(x|\alpha, \xi, \lambda) = M(x|\alpha, \xi) + \lambda P(x|\alpha, \xi)$, where $\mu = 1.5369, 1, 0, -2, -5, -10, \lambda = -1.2155, -1(1)2, 5, 10$ and in both cases $\alpha = 1.069, \xi = .765, x = [0(.1)2, 2.2, 2.5, 3; 3D]$.

<div align="right">*Extracts from text*</div>

<div align="center">MATHEMATICAL TABLES—ERRATA</div>

References have been made to Errata in RMT **477** (Cazzola), **484** (NBSCL), **486** (Anan'ev), **495** (Chebyshev, Whittaker & Robinson), **503** (Magnus & Oberhettinger), **506** (U. S. Navy), **510** (Skinner); N **87** (German Tables), **92** (Fontoura & Penteado).

124. J. R. AIREY, "Tables of the Bessel functions $J_n(x)$," $n = 0(1)13$, $x = [6.5(.5)16; 10D]$, BAAS, *Report*, 1915, p. 30–32.

We note the following 51 cases, for $n = 2(1)13$, where Airey and Harvard (RMT 380, 440, 501) differ:

n	x	Airey	Harvard
2	6.5	−0.30743 02368	−0.30743 03906 30...
3		−0.03534 65366	−0.03534 66312 85..
4		+0.27480 26645	+0.27480 27310
5		0.37356 52006	0.37356 53771
6		0.29991 30288	0.29991 32338
7		0.18012 03909	0.18012 05930
8		0.08803 85825	0.08803 88126
9		0.03658 99659	0.03659 03304
10		0.01328 74770	0.01328 82562
11		0.00429 45787	0.00429 66118
12		0.00124 80202	0.00125 41220
13		−0.00031 35057	+0.00033 39927
4	8.5	−0.20767 73541	−0.20770 08835
5		+0.06715 51647	+0.06713 30194
6		0.28668 34302	0.28668 09063
7		0.33757 43838	0.33759 29660
8		0.26932 14373	0.26935 45671
9		0.16938 36158	0.16942 73956
10		0.08937 32784	0.08943 28589
11		0.04090 64511	0.04100 28606
12		0.01650 22421	0.01669 21921
13		0.00568 81149	0.00612 80346
11	9.5	0.08969 64138	0.08969 64137
12		0.04269 16061	0.04269 16060
13		0.01815 60647	0.01815 60646
6	11.5	−0.24508 38970	−0.24508 14040
7		−0.08462 70668	−0.08462 44653
8		+0.14205 96418	+0.14206 03158
9		0.28227 52641	0.28227 36003
10		0.29976 25107	0.29975 92326
11		0.23905 08414	0.23904 68041
12		0.15755 21425	0.15754 76971
13		0.08975 36298	0.08974 83898
8	13.5	−0.20367 10209	−0.20367 08728
9		−0.02727 91962	−0.02727 93539
10		+0.16729 87593	+0.16729 84008
11		0.27512 92100	0.27512 88367
12		0.28105 99533	0.28105 97034
13		0.22453 29292	0.22453 28582
9	14	−0.11430 71982	−0.11430 71981
3	14.5	−0.21021 97923	−0.21021 97924 22...
4		−0.02612 20608	−0.02612 25583
5		+0.19580 76209	+0.19580 73465
6		0.16116 17993	0.16116 21076
7		−0.06243 23387	−0.06243 18091
8		−0.22144 12987	−0.22144 10957
9		−0.18191 66806	−0.18191 69861
10		−0.00438 63048	−0.00438 68871
11		+0.17586 66050	+0.17586 61074
12		0.27121 83952	0.27121 82225
13		0.27304 66008	0.27304 68125

R. C. A.

The discrepancies indicated above have all been tested by comparison with the original calculations, performed under the supervision of L. J. COMRIE, for the forthcoming BAAS, *Mathematical Tables, Bessel Functions*, part II. In every case the Harvard value is confirmed, although incompletely in 2 cases. These two cases are:

(i) $J_9(14)$ where Comrie's 12-decimal value ends ...81 50. The value given by Meissel (see GRAY, MATHEWS & MACROBERT *A Treatise on Bessel Functions*) has ...81 497..., so that Airey's error may presumably be explained as a rounding-off to 12 decimals, followed by another rounding-off to 10 decimals without reference back to Meissel's table, which Airey quotes as his source for integer x. This error is, however, quite trivial.

(ii) $J_9(14.5)$. The B.A. 12-decimal value ends $\ldots 61\ 51\ldots$, but is subject to a possible error of 2 units or so in the last figure. It has not seemed worthwhile to pursue the matter further.

It is disconcerting to find serious errors of this nature in any work of Airey's, even though it is comparatively early work. It is thus desirable to investigate more closely. A large pile of Airey's manuscripts were handed over to Dr. Comrie after Airey's death in 1937, but the calculations for this particular table do not seem to have been included.

The published values were therefore tested by formation of the values of

$$E = xJ_{n-1}(x) - 2nJ_n(x) + xJ_{n+1}(x)$$

in which $J_n(x)$ is used to denote Airey's tabulated value.

The following values are seriously in error.

x	n	E in units of the 10th decimal
6.5	1	$+\ \ \ 10000$
	12	$+\ \ \ \ \ 638$
8.5	3	$+20\ 00002$
11.5	5	$-\ 2\ 86690$
13.5	7	$-\ \ \ 19996$
	8	$+\ \ \ 44998$
14.5	3	$+\ \ \ 72132$

Apart from these the greatest value of $|E|$ is 18 units for $x = 14.5$, $n = 2$, arising from the small error at $n = 3$.

Thus most of Airey's errors are readily explained. All of them, except the end figure ones, arise from the major computation errors indicated in the list. Four of these errors (the 1st, 3rd, 5th and 6th) are almost certainly accounted for as errors in addition, subtraction or transcription. The second seems to fall into the same class, since $638 = 650 - 12 = 100x - 12$, and 12 units of the 10th decimal is a residual that might reasonably arise from rounding-off errors. The cause of the other two is, however, obscure.

It therefore appears probable that the errors have arisen through the publication of an early work without applying any check, since any check must surely have brought to light at least one of these errors, and so have directed Airey's attention to the need for a thorough examination. It is unfortunate that he did not carry the values for $x = 6.5$ a little beyond $n = 13$, for the error is within 4 or 5 lines of taking complete charge, and swamping the true value of $J_n(x)$ entirely.

J. C. P. MILLER

1 January 1948

EDITORIAL NOTE: There are 49 titles (1911–1938) in the list of published mathematical tables by J. R. AIREY (1868–1937). The title here in question is no. 11. Dr. Miller's report on Errata in the 6D portion of this table will appear in *MTAC* 23.

125. ALBERT GLODEN, "Table de factorisation des Nombres $X^4 + 1$ dans l'intervalle $1000 < X \leq 3000$"; see RMT 348, *MTAC*, v. 2, p. 211.

Corrections of five errors in this table are as follows:

P. 73 $1120^4 + 1 = 17 \times 89 \times 1\ 039\ 999\ 577$,
p. 82 $2310^4 + 1 = 17 \times 41 \times 40\ 852\ 171\ 033$,
p. 76 $\frac{1}{2}(1417^4 + 1) = 90\ 841 \times 22\ 190\ 521$,
p. 77 $\frac{1}{2}(1623^4 + 1) = 17 \times 204\ 077\ 517\ 313$,
p. 82 $\frac{1}{2}(2313^4 + 1) = 433 \times 593 \times 55\ 735\ 249$.

17 of the blank spaces in the table may now be filled in as follows:

P. 74 $1140^4 + 1 = 592\ 649 \times 2\ 849\ 849$,
p. 74 $1242^4 + 1 = 565\ 921 \times 4\ 204\ 657$,
p. 75 $1354^4 + 1 = 593\ 081 \times 5\ 667\ 097$,
p. 76 $1444^4 + 1 = 539\ 089 \times 8\ 065\ 073$,

p. 79 $1904^4 + 1 = 595\ 201 \times 22\ 080\ 257,$
p. 82 $2190^4 + 1 = 530\ 713 \times 43\ 342\ 777,$
p. 83 $2332^4 + 1 = 518\ 417 \times 57\ 047\ 281,$
p. 83 $2406^4 + 1 = 17 \times 511\ 457 \times 3\ 854\ 113,$
p. 74 $\frac{1}{2}(1229^4 + 1) = 597\ 769 \times 1\ 908\ 279,$
p. 75 $\frac{1}{2}(1299^4 + 1) = 527\ 377 \times 2\ 699\ 513,$
p. 79 $\frac{1}{2}(1867^4 + 1) = 563\ 081 \times 10\ 788\ 881,$
p. 79 $\frac{1}{2}(1869^4 + 1) = 522\ 553 \times 11\ 675\ 537,$
p. 80 $\frac{1}{2}(2005^4 + 1) = 2089 \times 3\ 868\ 023\ 217,$
p. 80 $\frac{1}{2}(2055^4 + 1) = 17 \times 572\ 233 \times 916\ 633,$
p. 82 $\frac{1}{2}(2189^4 + 1) = 526\ 249 \times 21\ 815\ 329,$
p. 87 $\frac{1}{2}(2895^4 + 1) = 570\ 001 \times 61\ 615\ 313,$
p. 87 $\frac{1}{2}(2969^4 + 1) = 520\ 529 \times 74\ 639\ 009.$

Furthermore, besides the factors given at the following 17 entries, the remaining factor in each case is a 12-figure prime of the form $8k + 1 < 600\ 000^2$: (a) $X^4 + 1$, p. 77, $X = 1562$; p. 79, $X = 1818$; p. 79, $X = 1848$; p. 82, $X = 2262$; p. 82, $X = 2302$; p. 84, $X = 2468$; p. 84, $X = 2476$; p. 86, $X = 2808$; p. 88, $X = 3000$. (b) $\frac{1}{2}(X^4 + 1)$, p. 78, $X = 1709$; p. 78, $X = 1715$; p. 82, $X = 2211$; p. 82, $X = 2299$; p. 84, $X = 2533$; p. 84, $X = 2577$; p. 85, $X = 2669$; p. 85, $X = 2683$.

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126. O. A. Walther, "Bemerkungen über das Tschebyscheffsche Verfahren zur numerischen Integration," *Skandinavisk Aktuarietidskrift*, v. 13, 1930, p. 168–192.

On p. 177–179 are given the roots of Chebyshev's polynomials of degree n (those employed in numerical integration with equal weight factors) for $n = [1(1)7; 10D]$, $n = [8(1)-10; 5D]$. This fact was unknown to the writer when he also gave, among other quantities, the roots, to 10D, for those polynomials having only real roots (see RMT **495**). The writer had relied on FMR, *Index*, p. 360, where no mention was made of Walther's roots which supersede the calculations of most of the authors cited there; hence the writer's statement on p. 192–193 regarding other tables of roots must be slightly modified in order to take Walther's tables into account.

Comparison of Walther's roots with the writer's, where they overlapped, revealed one appreciable error in Walther's calculations which was greater than a unit in the last place. On p. 178, for Walther's $n = 4$ (corresponding to the writer's $n = 5$) a pair of roots is given as $\pm 0.83249\ 74841$, whereas it should have been $\pm 0.83249\ 74870$.

Herbert E. Salzer

127. Dov Yarden, (a) "Table of Fibonacci numbers"; (b) "Table of the ranks of apparition in Fibonacci's sequence," *Riveon Lematematika*, v. 1, 1946, p. 35–37; 54; v. 2, Sept. 1947, p. 22. The errata below supplement those listed in *MTAC*, v. 2, p. 343–344.

(a) In the last number (5) of *Riv. Lem.*, v. 1, June 1947, p. 99, are following corrections in factorizations of U_n and V_n:

	U_n	n
$5 \cdot 28657 \cdot (33720414042782577761)$	$4831629526120101632848885$	115
$353 \cdot 709 \cdot 8969 \cdot 336419 \cdot 2710260697$	$20467111114739846236917559$	118

	V_n	$*$
$3 \cdot 347 \cdot 1270083883$	1322157322203	58
$2^2 \cdot 19 \cdot 199 \cdot 991 \cdot 2179 \cdot 9901 \cdot 1513909$	489526700523968661124	99
$2 \cdot 3^2 \cdot 227 \cdot 29134601 \cdot (5608975608563)$	66771477840504325965 1218	114
$2^2 \cdot 19 \cdot 79 \cdot 521 \cdot 859 \cdot (1052645985555841)$	282848519090497185 3895196	117

(b) On p. 22 of v. 2, line 13, the author noted the following corrections:

p = 1031, *for* $2 \cdot 5 \cdot 103$ 1030, *read* $2 \cdot 5 \cdot 103$ 206;

p = 1231, *for* $2 \cdot 3 \cdot 5 \cdot 41$ 1230, *read* $2 \cdot 3 \cdot 5 \cdot 41$ 410.

UNPUBLISHED MATHEMATICAL TABLES

Reference has also been made to Unpublished Tables in RMT **485** (Glaisher), **491** (Gloden); Q24 (Wrench).

67[F].—P. POULET, "Suites de totalics au depart de $n \le 2000$." Hectographed copy on one side of each of 56 leaves, in possession of D. H. L. 20 × 24.8 cm.

By a "totalic series" or "aliquot series" is meant a sequence of positive integers, each term of which is the sum of the proper divisors of its predecessor. Two simple examples are

18, 21, 11, 1

1420, 1604, 1210, 1184, 1210, 1184, ⋯.

The first of these terminates with its fourth term; the second ultimately becomes periodic of period two. It has been conjectured[1] that aliquot series either terminate or become periodic. The present tables show this to be the case for all such sequences whose "leaders" (first terms) do not exceed 2000, with the possible exception of about 25 series which are left unfinished. For each such leader are given those terms of the sequence which are $\le n$. When a term n_1 finally falls below n, the reader is referred to the previous series whose leader is n_1. When the leader is a term of a previous series, reference is made to the leader of this series. Prime leaders are of no interest and are omitted. Beyond $n = 200$ only abundant leaders n are listed. Other leaders would have given second terms not greater than the leaders.

Some leaders generate unusually long sequences. The longest completed series is

936, 1794, 2238, 2250, ⋯, 74, 40, 50, 43, 1

and runs to 189 terms, the largest term being

3328 91620 99526 = $2 \cdot 25943 \cdot 641582741$.

Thus the three dots of this series represent a formidable calculation. It is due to B. H. BROWN who (since 1940) also contributed many terms to several of the other still incomplete series. The incomplete series with the smallest leader is

276, 396, 696, 1104, ⋯, 5641400009252, ⋯ (58 terms).

Besides giving the terms in their decimal representation, the author gives their canonical factorization into primes. This table is an extension of a previous table of DICKSON[2] for leaders ≤ 1000.

D. H. L.

[1] L. E. DICKSON, *History of the Theory of Numbers*, v. 1, Washington, Carnegie Institution, 1919; offset print, New York, Stechert, 1934, p. 48–49.

[2] L. E. DICKSON, "Theorems and tables on the sum of the divisors of a number," *Quart. Jn. Math.*, v. 44, 1913, p. 267–272. For additions and corrections see P. POULET, *La Chasse aux nombres*, Brussels, v. 1, 1929, p. 69–72; v. 2, 1934, p. 187–8.

68[G].—HERBERT E. SALZER, *Chebyshev Polynomials*, ms. in possession of the author at NBSCL.

C. LANCZOS, in his "Trigonometric interpolation of empirical and analytical functions," *Jn. Math. Phys.*, v. 17, 1938, p. 140, gave the coefficients of the Chebyshev polynomials $C_n(x)$ adjusted to the range $[0, 1]$, up to $n = 10$. Due to their importance, these coefficients

were extended by the writer up to $n = 20$. Incidentally this also gives the numerical values of the coefficients of $C_n(x)$, range $[-1, 1]$, for n even, up to $n = 40$. For the coefficients of $C_n(x)$, range $[-1, 1]$, up to $n = 20$, see JONES, MILLER, CONN, & PANKHURST, R. Soc. Edinb., *Proc.*, v. 62A, p. 190 (*MTAC*, v. 2, p. 262). For x^n in terms of $C_n(x)$, for either the range $[-1, 1]$ or $[0, 1]$, only binomial coefficients are needed (readily available up to $n = 50$ in J. W. L. GLAISHER, *Mess. Math.*, v. 47, 1917, p. 97–107).

H. E. SALZER

AUTOMATIC COMPUTING MACHINERY

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TECHNICAL DEVELOPMENTS

The leading article of this issue of *MTAC*, "A Bell Telephone Laboratories' Computing Machine—II," by Dr. FRANZ L. ALT, is our current contribution under this heading.

DISCUSSIONS

Applications of Large-Scale High-Speed Computing Machines to Statistical Work

This discussion is essentially the reproduction of a talk given by Mr. J. L. McPHERSON of the Census Bureau, Washington, D. C. See under NEWS.

The construction and use of high-speed computing machines is a comparatively new art. This art is old enough, however, to have developed some specialized meanings for certain words. The terms defined in the following short glossary are used in their technical sense in this discussion.

1. Memory: A device into which code can be entered, and then abstracted at a later time.
2. Word: A group of digits (usually the equivalent of 10 or 12 decimal digits) stored in coded form in a single memory position.
3. Memory Position: One of N possible positions which a word may occupy in the memory.
4. Message: A group of words, usually that group of words required to describe one statistical observation.
5. Instruction: A word directing the machine to perform a particular operation.
6. Program: A series of instructions directing a sequence of operations.

The very newness of the art makes it rather difficult to talk about statistical applications for high-speed computing machines. At present, there are but a few such machines in existence. To date, these machines have been used on problems not particularly representative of the statistician's work. Therefore, it must be kept in mind that these remarks refer to proposed machines. The features and characteristics discussed should be interpreted as performance specifications. Competent experts at the National Bureau of Standards are optimistic about the possibility of building machines to meet these specifications. However, a sound knowledge of the statistical applications of such machines must be based on actual use. Accurate and detailed

descriptions of statistical applications of high-speed computing machines
await three events. First, a machine must become available to a group en-
gaged in large-scale statistical activity. Second, that group must become
skillful in the use of the machine. Third, a comprehensive series of tests must
be performed.

The words "high-speed" in the phrase "large-scale high-speed computing
machine," when translated into operations per unit of time, are what in-
terest most of us when we are introduced to these machines. It has become
rather common in this field to use multiplication time as an over-all measure
of the speed of operation of a machine. If we accept this convention, we
cannot fail to agree that the machines are "high-speed" indeed when we are
told that they will determine the product of two ten-digit decimal numbers
in about two milli-seconds. These tremendously high speeds are possible
because all calculations are accomplished by controlling the behavior of
electrical pulses that can be generated and dispatched through the various
circuits of the machine at rates of a million or more pulses per second.

The practice of expressing machine speed in terms of multiplication time
originated with mathematicians, physicists, and engineers whose interest
was in the development of equipment capable of providing numerical solu-
tions according to complicated and involved formulae. From their experi-
ence with such formulae, they knew that in every 100 arithmetic operations
there would be about 65 additions or subtractions, about 32 multiplications,
and about three divisions. Since algebraic addition was most frequent but
could be performed much faster than multiplication, while division, although
slower than multiplication, occurred far less frequently, the selection of
multiplication time as a measure of machine speed was a logical and in-
telligent choice.

Much of the statistician's work, however, differs significantly from the
work of other scientists. Whereas the mathematician or physicist is con-
fronted with a tremendous amount of arithmetic manipulation of a com-
paratively *small* volume of original data, frequently the statistician re-
quires only a moderate amount of arithmetic manipulation of a tremendously
large volume of original data.

This is a significant and important point. The largest of the high-speed
machines conceived to date will have an internal memory of a few thousand
words at most. The statistician will usually require several words for each
message. It is, therefore, readily apparent that statistical studies involving
thousands, or hundreds of thousands, or millions of cases will involve a
number of "words" many times the capacity of the internal memory of one
of these machines. The benefits of high-speed computation will be largely
lost to the statistician unless there is associated with the machines some
means whereby information can be delivered to the machine from the out-
side, and transmitted by the machine to the outside, at speeds comparable
to, or at least in balance with, the internal speed of operation of the machine.
In other words, the statistician demands high-speed input and output for a
large and important area of his work. I believe the staff at the Bureau of the
Census, in early discussions with designers and builders of these machines,
were the first to call this extremely important requirement to their attention.
It was pointed out that a convention that measured machine speeds in
terms of the time required to perform a multiplication would be acceptable

only if input and output speeds were in balance with the computing speeds of the machine.

The engineers were equal to this challenge. They have designed machines that will do a moderate amount of work on a very large amount of input data at speeds of the same order of magnitude as those that will be attained on problems involving a large amount of work on a small amount of input data. In other words, they have very nearly solved the problem of high-speed input and output.

Metal wire or tape will be used on some, if not all, of the high-speed electronic machines as both an input and an output medium. Data will be recorded magnetically on these tapes and the machines will be able to read from, or write on, the tapes. To perform either of these operations, it will be necessary to move the tape physically. Since this mechanical movement of the tape will be involved, the speeds at which words can be read (or written) by the machine will be much less than the speeds at which the machine can transfer words internally. Here, however, the large internal memory of 1000 or more words comes to the rescue.

The machine will not read words from the tape one at a time as they are needed. Instead, it will read a block of words or several blocks of words from the tape, and store them in the internal high-speed memory to be examined and processed one word, or one message, at a time. Thus a portion of the internal memory will serve as a reservoir of unprocessed data. As the reservoir becomes depleted (but before it is exhausted), additional data can be read from the tape. This process of keeping a supply of data stored in the machine will be independent of the computing elements of the instrument. Thus, except for the time it takes to introduce data to the machine at the beginning of a problem, the input speed will be in balance with the internal speed of the machine. Since writing on a tape is simply the inverse of the reading process, the output speeds will also be in balance.

Now that we know the machines can accept vast quantities of data, perform certain arithmetic operations on those data, and deliver the results of those computations, all at very high rates of speed, we can explore the usefulness of the machines to statisticians.

Before investigating any specific types of statistical problems, however, I would like to discuss briefly a few of the properties and principles of operation of these machines.

The following diagram may help one to understand the logical organization of these machines.

High-Speed Bus

in	in	out	in	out	in	out	out
INPUT	MEMORY		ARITHMETIC UNIT		CONTROL		OUTPUT
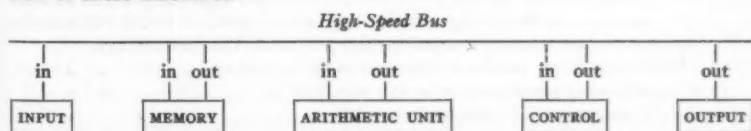
It is important to note that the control has associated with it an "in gate" as well as an "out gate." That is, words can enter the control in the same manner that they can enter the memory or the arithmetic unit. *The instructions governing the routine of operations that the machine is to perform can be stored in the memory in exactly the same manner in which the numbers on which the machine is to operate are stored.* This means that the program for

the solution of a problem can be recorded on magnetic tape and delivered to the machine via magnetic tape just as numbers are delivered to the machine. The length of the sequence of instructions can be flexible. If the problem involves only a short program of instructions, a comparatively short length of tape will be required. If a long routine of instructions is necessary, a longer length of tape will be required. Furthermore, the total number of instruction words may well exceed the capacity of the internal memory of the machine. The point is: not all of the instructions need be stored internally any more than all of the numbers need be stored internally. When additional instructions are required, they can be read into the machine from an instruction tape, just as additional numbers are read into the machine from a data tape when they are needed.

Another important characteristic of these machines is also clarified by this diagram. Since the instructions are stored in the memory along with numbers, the instruction words can be sent to the arithmetic unit just as number words can be sent to the arithmetic unit. *This means that the machine can generate or modify instructions to itself.*

It will not be necessary for the user of one of these machines to specify in his program a routine of instructions for each process each time that process occurs. All the programmer has to provide is instructions for a typical routine, and the machine will modify those instructions as the solution of the problem proceeds. For example, suppose we are obtaining a simple count of the number of whites and the number of nonwhites in a given population. Let us further suppose that each time we read from the data tape we provide the internal memory of the machine with the unclassified information for 50 persons. We, in effect, give the machine a sequence of instructions like this:

   a Bring information for first 50 persons into the high-speed memory from the data tape and put it in memory positions 1 to 50.

   b Bring information for the next 50 persons into the high-speed memory from the data tape and put it in memory positions 51 to 100.

   c Examine race in memory position [1]. (The brackets indicate the number they bound will change.)

   d Add "1" in the appropriate memory position (for example in memory position 201 if white, 202 if nonwhite).

   e Add "1" to the memory position specified in *c*.

   f Compare memory position number in $c = 51$, bring information for the next 50 persons into the high-speed memory from the data tape and put it in memory positions 1 to 50 (putting new data into memory positions 1 to 50 will erase the information previously stored in those positions). Then proceed to *g*.

      2. If the memory position number in $c \neq 51$, proceed to *g*.

   g Compare memory position number in *c* with "101":

      1. If memory position number in $c = 101$,

         (a) bring information for next 50 persons into the high-speed memory from the data tape and put it in memory positions 51 to 100; then

         (b) subtract 100 from the memory position number in *c*; then

         (c) go back to *c*.

      2. If memory position number in $c \neq 101$, go back to *c*.

The instructions in *f* 1 and *g* 1 (a) are tape-reading instructions. As previously indicated, the machines will execute tape reading (or writing)

instructions in parallel with the execution of other instructions. Thus the above sequence shows how a reserve of unprocessed data can be left in the internal memory as well as how the machine modifies its own instructions. There are many other ways in which the machine's ability to modify its own instructions is useful. This illustration is intended only to suggest the power of this feature.

The manner in which these machines read and write words should also be mentioned. In general, the process of "reading" a word does not erase that word. Only when a word is to be "written" in the space occupied by another word is the original word erased. This is true of the high-speed internal memory and usually of the magnetic tapes. (We can, if we wish, specify that after it is read a tape shall be erased.) Thus a tape containing the instructions for a recurring process (a monthly tabulation, for example) need be prepared only once. It can be read and reread as often as desired. On the other hand, in the course of rearranging data the machine may write intermediate results on tapes and leave them so written for only a short period, after which other results will be written on the same tapes.

Now let us examine some of the ways in which these machines may be useful to statisticians. First, and probably most obvious, is the reduction in the amount of preliminary rearrangement that will be required in the process of organizing and classifying data. Expressing this thought in the terminology of punch card tabulation, this means there will be a significant reduction of the volume of "sorting" necessary to prepare a tabulation. For example, assume a table with 20 columns and 25 lines—a 500-cell table. Such a table is not large, nor is it especially small. To prepare this table using punch cards, we would sort the cards into the groups representing the 25 lines and then run them through a tabulator to obtain the 20 columns on the other axis of the table. Now, suppose we have a high-speed computing machine with 1000 memory positions. We can make the very liberal allowance of 500 of these memory positions for the storage of instructions and unprocessed data and still have 500 memory positions in which to accumulate the 500 numbers we need for the table. In this case, it will be entirely unnecessary to rearrange the unprocessed data. We can leave it in whatever arrangement it appears on the data tape (random or otherwise), and in one passage of the tape through the machine we will obtain the desired table. Of course, for many tabulations some rearrangement of data will be necessary, but obviously the large number of memory positions in the proposed machines will effect substantial reductions in the number of "sort groups" that must be established to prepare a given tabulation.

Perhaps I can best illustrate other ways in which these machines will be useful by compounding the requirements associated with the 500-cell table. For the sake of illustration, assume that the 20-category classification refers to height and the 25-category classification refers to weight. We may also assume a machine with four tapes and their associated reading, writing, and driving mechanisms. Assume that the program is recorded on tape 1 and the data (say several thousand observations for each of which we have a height and a weight) are recorded on tape 2. Assume further that in addition to height and weight we also have the age of each individual recorded on tape 2. Age is not a variable in the table that we are preparing, but for future use it would be desirable to have the data arranged in order of age.

We can have on the first section of the program tape (i.e., tape 1) the program for arranging the data in order by age. This will involve transferring data between tapes 2, 3, and 4 until we have them in order of age, say, back again on tape 2.

Next, we can read from tape 1 (the instruction tape) the program for preparing the height-by-weight table. Let us suppose that the execution of these instructions terminates in recording, on tape 3, the 500 numbers corresponding to the 500 cells of the table.

We can now bring in some more instructions from tape 1. These might be a routine of orders for the computation of percent distributions. If so, they might direct the machine (a) to read the numbers from tape 3 (on which the 500 numbers in our table are recorded), (b) to accumulate the appropriate totals (for example, the total for each of the 20 columns), (c) to use those totals as bases and compute the desired percentages, and (d) to record the 500 percentages on tape 3.

Upon the completion of this routine, we can require still more instructions to be read from tape 1. These might be a program of orders directing the calculation of various statistical constants describing our distribution. These measures might include the mean height and mean weight, the variance of each variable, the covariance, the various correlation coefficients and the regression coefficients.

This example is representative of the type of work frequently done by statisticians. Facilities now available can provide all of the results described as obtainable by using a high-speed computing machine. How will the high-speed computing machines be superior to existing equipment? Obviously, we expect them to be much faster than anything now available to us. Suppose the work described in the illustration was for a study involving 100,000 observations. A high-speed computing machine should complete the job in a few hours at the most. Another and equally important contribution of these machines is their automatic nature. In the above illustration, the instructions on tape 1 directed the machine to perform a variety of operations. Data were rearranged, a table was prepared, percentages and other statistical measures were computed. All that was necessary to accomplish all of these objectives was the placing of tapes 1, 2, 3, and 4 on the machine and pressing a start button. Thereafter, the machine took over. This characterizes a major way in which these machines will help statisticians. Obviously, the incidence of human errors will be greatly reduced by a machine which eliminates human operators at so many stages of the tabulation and calculation processes.

BIBLIOGRAPHY Z–III

1. ANON., "Electronic computer assures solution of scientific problems," *Iron Age*, v. 157, 28 Feb. 1946, p. 132–134, 1 illustr. 22.8 × 29.2 cm.
2. ANON., "Electronic computer known as the ENIAC," *Mechanical Engineering*, v. 68, 1946, p. 560–561. 21.6 × 27.9 cm.
3. ANON., "Mathematics by robot," *Army Ordnance*, v. 30, 1946, p. 329–331, 4 illustrs. 21.6 × 27.9 cm.

A description of the ENIAC mentioning its purpose, size, phenomenal speed, operational characteristics, necessity for programming problems to be put on the machine, and future trends in electronic computing.

4. ANON., "War Department unveils 18,000-tube robot calculator," *Electronics*, v. 19, April 1946, p. 308–314, 3 illustrs. 20.3 × 29.8 cm.

An article dealing with ballistic applications, industrial uses, general details, operating procedure, arithmetic elements, memory elements, and control elements of the ENIAC.

5. O. CESAREO, "The relay interpolator," *Bell Laboratories' Record*, v. 24, 1946, p. 457–460, 1 illustration, 2 circuit diagrams. 25.4 × 17.8 cm.

Presented here is a brief description of the first all-relay digital computer, suggested by G. R. STIBITZ for use in gun director test equipment and developed by the Bell Telephone Laboratories for the National Defense Research Committee. The interpolator computations are based on the use of a bi-quinary adder and register circuits. These circuits are explained briefly. Block schematics and circuit diagrams showing inter-connection of contacts of the adder relays are included. The interpolator makes use of 493 U-type relays. Now in use at the Naval Research Laboratory, it was given to the Navy by the National Defense Research Committee.

6. JOSEPH JULEY, "The ballistic computer," *Bell Laboratories' Record*, v. 25, 1947, p. 5–9, 3 illustrations, 2 tables, 1 block layout. 25.4 × 17.8 cm.

The Ballistic Computer, like the Relay Interpolator (see above), was designed for the testing of gun directors. Director operators track a plan simulating a bombing run while director indicators and gun order indicators are photographed at regular intervals. With these data, transferred by key-punch operators onto teletype tapes, the computer calculates, for a successive series of instants, exactly where the plane and the shell would have been at the time of shell burst. The distance between the positions of shell and plane at each instant is the gun director error, given in errors in range, azimuth and angle of elevation.

The Ballistic Computer differs from the Relay Interpolator, its forerunner, in having more relays—1300 as compared to 493—and in having, in addition to register relays for adding, a set of multicontact relays used in multiplying and dividing. The computer can run continuously on one tape loading for 24 hours or longer. Useful checking circuits have been incorporated into the machine, making it quite feasible to run it unattended over long periods. The Ballistic Computer can accomplish, in five or six hours, work that five men, working steadily, could not complete in less than a week.

7. ANON., "The differential analyzer" [at Manchester University], *The Engineer*, London, v. 160, 1935, p. 56–58, 82–84, 12 illustrations. 24.3 × 36.8 cm.

This article describes the mechanical integrator, input table, torque amplifier, 2-stage torque amplifier, reversible torque amplifier, and frontlash unit.

8. D. R. HARTREE, "Differential analyzer," *Nature*, v. 135, 1935, p. 940–943, 2 illustrations. 18.4 × 27.3 cm.

This article discusses the construction, operation, and applications of the differential analyzer at Manchester University. It also treats the development of the torque amplifier by DR. VANNEVAR BUSH as then used at the Massachusetts Institute of Technology.

In earlier pages of *MTAC* there have been numerous references to Differential Analyzers: v. 1, p. 62–64, 96–97, 370, 430–431, 452–454 (Bibliography of 29 titles); v. 2, p. 55, 65, 89–91, 115–117, 150, 282, 293, 371. The first Differential Analyzer was described by the Russian A. KRYLOV in "Sur un intégrateur des équations différentielles ordinaires," 7 illustrations, Akad. N., S.S.S.R., *Bulletin*, s. 5, v. 20, Jan. 1904, p. 17–37.

9. ARTHUR W. BURKS, HERMAN H. GOLDSTINE, & JOHN VON NEUMANN, *Preliminary Discussion of the Logical Design of an Electronic Computing Instrument.* Second ed., part 1, v. 1, Princeton, N. J., Institute for Advanced Study, 1947, vi, 42 leaves. 21.6 × 27.9 cm. See MTAC, v. 3, p. 50–53, for a review of the first edition.

As the authors state in the preface, the arithmetic organ has been discussed in greater detail and the arithmetic processes treated more completely in this edition. In addition, certain sections of the report have been made somewhat more specific in the light of engineering advances in the authors' laboratory.

The treatment of the arithmetic operations on numbers expressed in binary form is more satisfying than the corresponding part of the first edition. A discussion of the choice of location of the binary point has been added. The arithmetic operations are carefully discussed in detail from the standpoint of adaptation to their execution by electronic organs. A few numerical examples provide useful illustrations for the reader. The treatment of the control organ is somewhat more closely integrated with the discussion of the arithmetic operations.

This second edition of the report is of interest in that it presents the most recent ideas of a competent group of designers of an automatically-sequenced, electronic digital computing machine. The format has been improved, affording more pleasant reading than the mimeographed first edition. Unfortunately, however, the report is marred by typographic errors which tend to confuse the reader. A list of errata would be a useful and welcome addition.

MDL

10. MOORE SCHOOL OF ELECTRICAL ENGINEERING, University of Pennsylvania, *Theory and Techniques for Design of Electronic Digital Computers.* Lectures delivered 8 July–31 August 1946. Two volumes, Philadelphia, The University of Pennsylvania, v. 1 (lectures 1–10) publ. 10 Sept. 1947, 161 leaves; v. 2 (lectures 11–21) publ. 1 Nov. 1947, 173 leaves. 21.5 × 27.7 cm. Offset printing of typescript on one side of each leaf.

These volumes were reviewed by Dr. C. H. PAGE & Mr. S. N. ALEXANDER, both of the National Bureau of Standards, Washington, D. C. Mr. Alexander summarized and criticized those chapters of the report which are concerned with machine design. Dr. Page, on the other hand, has limited his review to those chapters of the report dealing with the mathematical phases of electronic digital computers.

Introductory Comment.—The first two reports contain twelve lectures on machine design considerations that range from the philosophy and economics of automatic computation to details about the organizational and technical features of the machines. In general, the reviewer has found most of the material to be considerably improved over the original form in which he heard the lectures delivered at the University of Pennsylvania. In several instances the subject matter appears either to have been augmented or brought up to date.

Lecture 1. *Introduction to the Course on Electronic Digital Computers* by G. R. STIBITZ. This lecture provides an excellent background from which to view the present intensive activity being directed toward devising more powerful aids to computation. First, the motivations that have led men to devise such aids are reviewed. Out of this the predominantly economic justification that underlies the current programs is convincingly demonstrated. Some of the areas for future application of automatically sequenced digital computers are extrapolated from the limited existing experience in their use. However, the significant point is made that, as in other fields, we can expect a widening of the market in keeping with further reductions in the unit cost of computation. Certainly the salient feature of this lecture is the interesting manner in which the thread of economic justification is woven into the discussion of the computers.

Lecture 2. *The History of Computing Devices* by I. A. TRAVIS. This lecture gives a concise history of aids to computations and some of the circumstances that led to their invention and use. The present emphasis on the development of automatic digital machines is dated from 1938, with the prediction that this emphasis will continue during the next decade. An interesting chart is included, which enumerates what the lecturer considers to be the important dates in the growth of automatic computation.

Lecture 3. *Digital and Analogy Computing Machines* by J. W. MAUCHLY. The lecturer presents his thoughts regarding the appropriate fields of application for the newer digital machines as compared to the already well developed analogue devices. The analogue procedure for attaining numerical solutions is examined, and its characteristics and limitations are noted. Against this background the virtues of the digital procedure are pointed out. Paramount is the fact that the digital procedure offers a feasible attack on classes of problems not readily within the scope of analogue machines that have been devised thus far. The digital machines are further recommended for their inherently greater flexibility and their ability to attain greater accuracy whenever this accuracy is required. The lecturer then goes on to a discussion of the four steps into which he divides the over-all task of using a digital machine to obtain a solution: (1) coding, (2) set-up, (3) operations, (4) interpretation. The importance of balance among these steps is considered, particularly in relation to minimizing both the time and cost of obtaining a solution.

In closing, the lecturer covers in a very general manner the basic choice of the serial versus parallel organization of the machine itself, giving his views on the influence of this choice upon the speed, cost and reliability of the machine.

Lecture 8. *Digital Machine Functions* by A. H. BURKS. This lecture examines some of the factors that govern the advisability of having the digital computer always reduce the solution of each problem to first principles. The possibility of performing all the mathematical procedures in terms of four elementary logical operations is pointed out in a tantalizingly brief discussion. The editorial note on this topic, perhaps because of its excessive brevity, provides little clarification for a newcomer to such considerations. The lecturer then points out that practical considerations dictate the use of units at least capable of directly performing the arithmetic operations of addition and multiplication. The more elaborate operations such as finding roots, integration and sorting are better derived from the basic arithmetic operations through appropriate instructions to the machine. Most of the remaining text is devoted to methods of handling negative numbers and subtraction through the use of both the "nines" and "tens" complements. The special design considerations that are imposed on the adder and multiplier by each system of complements are also discussed. The concluding remarks point out how iterative procedures, based on the use of repeated addition and multiplication, can be employed to provide the operations of division and square root.

Lecture 9. *The Use of Function Tables with Computing Machines* by J. W. MAUCHLY. A rather detailed analysis of the general problem of referring to arbitrary functions is contained in this lecture. The complexity that arises for functions of several variables is brought out, and an elementary approach to this problem is cited. The relative merits of referring to the stored values of the functions in terms of a "selecting" or a "hunting" process are compared. Examples are given of arrangements where a combination of selecting and hunting is employed, together with a discussion showing the continuous transition that can occur between these two modes of reference to the function tables. Several examples are given of methods by which one can derive functional values between those tabulated within the machine. The discussion concludes with a few remarks on the importance of being able to provide extensive tabulations on the external memory of an EDVAC-type machine.

Lecture 10. *A Preview of a Digital Computing Machine* by J. P. ECKERT, JR. The design principles that are proposed for the EDVAC type of machine are discussed in the light of experience gained in the building and operation of the ENIAC. The major objections to the ENIAC are avoided in the EDVAC by (a) providing more memory, (b) separating the memory and arithmetic functions of the machine, (c) providing automatic setting up of the problem by the use of the internal control system, (d) use of a serial type of programming in order to simplify the planning of the problem, (e) the use of a more flexible type of input

and output equipment such as the erasable magnetic recording device. The lecture concludes with a discussion of the functional arrangement of the EDVAC-type machine, some of the components that might possibly be used to accomplish the operations desired, and an example of a code by which the machine could be instructed to carry out the sequence of operations.

Lecture 11. *Elements of a Complete Computing System* by C. B. SHEPPARD. The summary that has been prepared from this lecture is a highly condensed, but nevertheless lucid, account of several devices for realizing the functions desired in the principal units of the computing machine. The machine is divided into four units, which are designated as the high-speed memory, the computer, the control, and the input-output mechanism. The operating principles for each device selected for discussion are briefly and clearly presented. The section devoted to the high-speed memory is expanded to provide a condensed, yet readable, technical description for several of these devices. Indeed, this summary contains much of the flavor of an unabridged dictionary of digital computing machine terms.

Lecture 13. *The Automatic Sequence Controlled Calculator*, and Lecture 14. *Electro-Mechanical Tables of the Elementary Functions* by H. H. AIKEN. The salient features of the electro-mechanical calculator in service at Harvard University are concisely described. The internal organization of the machine is covered, including the means by which the sequence mechanism and its punched control tape direct the solution of the problem. Reference is made to the *Manual of Operation* (1946, *MTAC*, v. 2, p. 185–187, 368), where this material is given in detail. This lecture closes with a discussion of the modifications in equipment and machine operation that appear advisable in the light of service experience.

The subject of elementary functions begins with the derivation of reciprocal powers through an iterative procedure using repeated addition and multiplication. Such operations as division and square root can then be derived without need for specialized equipment. Even though the Harvard machine does have a divider unit, the iterative procedure has been used to obtain extra quotient precision. The next topic covers the principles used in the elementary function units of the machine. These units provide values of the logarithm, exponential and sine functions. The lecturer not only considers these specialized units to be desirable in a machine of this type, but expresses the need for a $\tan^{-1}$ unit, which is being designed.

Even though the techniques used in this first automatically sequenced digital calculator have little in common with the electronic machines, familiarity with its internal organization is useful background material for the design of any digital machine. Indeed, the experience derived from planning problems for this computer has been a useful guide to several groups now developing electronic machines.

Lecture 15. *Types of Circuits—General* by J. P. ECKERT, JR. The general design considerations for electron tube circuits, as employed in digital computers, are introduced by comparison to the more familiar relay equivalents. The lecturer gives an appraisal of the relative merits of electron tubes versus relays on the basis of speed, actuating power, cost, reliability and life. This is followed by a discussion of those characteristics of relays and electron tubes that relate to operation in the synchronous and nonsynchronous type of machines. The remainder of the lecture discusses several basic circuits out of which the arithmetic and control functions of the computer can be compounded. The treatment of the amplitude-sensitive class of circuit is so sketchy that the unaided newcomer will sense little beyond their existence and the lecturer's disapproval of their use.

Lecture 16. *Switching and Coupling Circuits* by T. K. SHARPLESS. This lecture presents a few of the more conventional circuits and devices having two stable states that are of interest in digital computing. These are used to develop the design principles used in the ENIAC decade counters. The next main topic is a discussion of several techniques and devices for performing the distribution function needed for the control of the computing system. This includes the separation of a time sequence of pulses into individual pulses, each on a separate circuit, and the inverse operation of assembling a spatial distribution of pulses on separate circuits into a time sequence on a single circuit. The concluding topic is a qualitative discussion of coupling circuits for amplifiers that have been used in computer

switching circuits. Emphasis is given to the departure from standard design permitted by the nonlinear character of the switching operations.

Lecture 20. *Reliability of Parts* by J. P. ECKERT, JR. This lecture gives an account of a number of the design choices made with respect to the ratings of parts that were used in the ENIAC. Much of it is then related to subsequent service experience. The surprisingly reliable service obtained from standard production types of electron tubes is of particular interest. (A recent summary of this experience is given in *Electronics*, v. 20, 1947, p. 116.) No doubt greater demands will be made on the electron tubes in future machines, but this initial experience is certainly encouraging. The quoted greater reliability of the electronic equipment in comparison to that of the standard punched card equipment associated with the ENIAC is a point of further significance to the future of this relatively new art.

Lecture 21. *Memory Devices* by C. B. SHEPPARD. After emphasizing the important role played by memory devices in the computer, the lecturer presents a table giving his appraisal of the relative merits of fourteen possible means of attaining useful memory. While the tabulation is an interesting one, it needs a fuller exposition of the sources and significance of the data than is provided. Next, some estimates are given regarding the minimum amounts of memory capacity that an electronic machine should contain. This is then divided into a high-speed main memory and a slow-speed auxiliary memory to conform to practical considerations imposed by available techniques. The case for magnetic wire and tape for the slow-speed memory is presented and is followed by a summary of the pertinent characteristics of these media. The remainder of the lecture is concerned with the high-speed memory problem. Although considerable attention is given to electrostatic types of memory tubes, the uninitiated reader may find the presentation somewhat confusing. Furthermore, he might obtain the erroneous impression that such tubes have emerged from the developmental laboratories. It is unfortunate that the subject of acoustic delay type of high-speed memory is disposed of by simply referring to Progress Reports on EDVAC of June 30, 1946. These reports have not been widely circulated, and the general reader is directed to a less complete account of this device given in *Electronics*, v. 20, Nov. 1947, p. 134.

The remaining nine lectures, on mathematical techniques involved in using digital machines, were as follows:

Lecture 4. *Computing Machines for Pure Mathematics* by D. H. LEHMER;
Lecture 5. *Some General Considerations in the Solution of Problems in Applied Mathematics*, by D. R. HARTREE;
Lectures 6, 7, 12, 17, 18. *Numerical Mathematical Methods*, 17 by A. W. BURKS, the other four by H. H. GOLDSTINE;
Lecture 10. *A Preview of a Digital Computing Machine* by J. P. ECKERT, JR.;
Lecture 19. *On the Accumulation of Errors in Numerical Integration of the ENIAC* by H. RADEMACHER.

Unfortunately, most of these were published in summary form only, reducing their utility to the mathematically minded engineer who is not familiar with the literature on computational methods.

The keynote is well expressed by Prof. HARTREE who made the point that developments in the design of mathematics for the machines must go hand in hand with the development of the machines for mathematics. Automatically sequenced machines, for example, are admirably adapted to iterative procedures, and the type of iterative algorithm most economically employed depends on the machine design, whether the machine employs a fixed time cycle or whether addition is performed more quickly than multiplication. Numerical procedures not useful on desk calculators, such as dividing by an iterative process involving only multiplication and addition, are often "natural" for machine operation. Variational and integral equation representations of problems may thus be more suitable than the customary procedures. Research in new computational methods is indicated, since most methods have been developed for use by human beings rather than by automatic machines.

Elementary processes of numerical quadrature and NEWTON-RAPHSON root evaluation are reviewed in several lectures. The iterative procedure for minimizing a function of several variables is derived in both algebraic and matrix notation, and the Steepest Descent and SOUTHWELL methods compared. The latter is simpler (and naturally more slowly convergent) for human use, but requires a machine to decide step-by-step which axial direction to follow in approaching the minimum of the function—a type of decision which may be costly to program into the machine. The minimization procedure is applied to a system of linear equations, and numerical examples given.

Ordinary differential equations with one-point boundary conditions can be integrated by an open-cycle process of successive low-order approximations applied separately to each subinterval of integration, or by single high-order approximations in each subinterval. The speed-accuracy compromise is different for human beings and machines, and may well differ among types of machines. The open-cycle method is discussed from the viewpoints of both the classical differential equation and the difference equation. Replacement of a single differential equation by a system of first order differential equations, and ultimately by a system of forward difference equations, is shown in an example to lead to an algorithm particularly adapted to machines.

The HEUN second order method of integrating differential equations, using an approximation corresponding to the trapezoidal rule in numerical quadrature, is analyzed for truncation error in the problem:

$$\frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y).$$

The round-off errors in this problem are analyzed for the simpler first-order approximation method, and as in the case of numerical quadrature, are shown to depend on $(\Delta t)^{-1}$ in the maximal appraisal, and on $(\Delta t)^{-\frac{1}{2}}$ in the statistical appraisal. The statistical treatment assumes random individual round-off errors and so requires sufficiently large Δt for validity. This Δt is evaluated as

$$(\Delta t)^2 |x'''| > 10^{-k}, \qquad (\Delta t)^2 |y'''| > 10^{-k},$$

where k is the number of digits accommodated by the machine. The statement is then made that "This is a restriction on the fineness of the step of integration. Actual tests show that this is a conservative estimate. . . ."

The conclusion appears unwarranted, since the restriction on Δt was not derived as a restriction involving computational accuracy, but only as a restriction on the method used to analyze this accuracy. The reviewer feels that there is a "no-man's land" of interval sizes just below the quoted limit for random round-off, but that for intervals sufficiently smaller, a type of uniformity appears which again allows error analysis and perhaps a round-off procedure with reduced error.

NEWS

British Association for the Advancement of Science.—At the annual meeting held at Dundee from August 27 to September 3, 1947, the following papers were presented during the discussion of modern methods of computation:

"General survey of computational methods" by Dr. J. C. P. MILLER.

"Principles of programming on large-scale calculating machines" by J. H. WILKINSON.

"Relaxation methods" by L. FOX.

"An electronic differential analyzer" by J. B. JACKSON.

Prof. R. V. SOUTHWELL, Imperial College, London, announced that he expects soon to perfect a practical method of relaxation for functions of three variables.

Association for Computing Machinery (formerly Eastern Association for Computing Machinery).—The Executive Committee met on October 24, 1947, at Columbia University, New York (see *MTAC*, v. 3, p. 57). The Committee now consists of:

Dr. JOHN H. CURTISS (President, A.C.M.), National Bureau of Standards, Washington, D. C.

Dr. JOHN W. MAUCHLY (Vice-President, A.C.M.), Electronic Control Company, Philadelphia, Pa.

Mr. E. C. BERKELEY (Secretary, A.C.M.), Prudential Insurance Company of America, Newark, N. J.

Mr. ROBERT V. D. CAMPBELL (Treasurer, A.C.M.), Raytheon Manufacturing Co., Waltham, Mass.

Dr. FRANZ L. ALT, Ballistic Research Laboratories, Aberdeen, Md.

Mr. E. G. ANDREWS, Bell Telephone Laboratories, New York, N. Y.

Mr. PERRY CRAWFORD, Special Devices Division, Office of Naval Research, Sands Point, L. I., N. Y.

Dr. GEORGE B. DANTZIG, Office of the Air Comptroller, Air Force, Washington, D. C.

Dr. JAN A. RAJCHMAN, R.C.A. Laboratories, Princeton, N. J.

Dr. MINA REES, Office of Naval Research, Washington, D. C.

Prof. JOHN B. RUSSELL, Columbia University, New York, N. Y.

Prof. RICHARD TAYLOR, Massachusetts Institute of Technology, Cambridge, Mass.

Dr. CHARLES B. TOMPKINS, Engineering Research Associates, Washington, D. C.

The second meeting of the Association was held at the Ballistic Research Laboratories, Aberdeen Proving Grounds, Aberdeen, Md., on Thursday and Friday, December 11 and 12, at the invitation of Col. LESLIE E. SIMON.

The Ballistic Research Laboratories, probably the largest computing laboratory of the world, has a unique collection of large-scale computing equipment. This includes the ENIAC, the first, and so far the only, large-scale electronic digital calculator; the "Bell Relay Computer," a large automatic installation designed by the Bell Telephone Laboratories; a new sequence-controlled relay machine of medium size, built by the IBM Corporation; and a differential analyzer. Demonstrations of these machines were given Thursday afternoon and Saturday morning.

On Thursday the following papers were presented:

(1) "General principles of coding, with application to the ENIAC" by J. VON NEUMANN, Institute for Advanced Study, Princeton, N. J.

(2) "Adaptation of the ENIAC to von Neumann's coding technique" by R. F. CLIPPINGER, Aberdeen Proving Ground.

(3) "Optimum size of automatic computers," by G. R. STIBITZ, Burlington, Vt. see *MTAC*, v. 2, p. 362–364.

The program for the Friday meeting was as follows:

(4) "The UNIVAC" by J. W. MAUCHLY, Electronic Control Company, Philadelphia, Pa.

(5) "The machine of the Raytheon Company" by R. V. D. CAMPBELL, Raytheon Co., Waltham, Mass.

(6) "Operating characteristics of the Aberdeen machines" by F. L. ALT, Aberdeen Proving Ground.

(7) "Reduction of DOPPLER observations" by D. HOFFLEIT, Aberdeen Proving Ground.

(8) "Partial differential equations" by B. L. HICKS, J. H. LEVIN, M. LOTKIN, R. F. CLIPPINGER, J. V. HOLBERTON, Aberdeen Proving Ground.

(9) "Application of large-scale high-speed computing machines to statistical work" by J. L. McPHERSON, Census Bureau, Washington, D. C.

The purpose of the Association is to advance the science, design, construction, and application of the new machinery for computing, reasoning, and other handling of information. The present membership comprises 246 individuals representing 80 organizations. Anyone interested in joining may become a member by sending his name, organization and address to the Secretary, Mr. EDMUND C. BERKELEY, Chief Research Consultant, the Prudential Insurance Company of America, Newark 1, N. J. Dues of $1.00 may be en-

closed with this information or mailed directly to the Treasurer, Mr. ROBERT V. D. CAMP-BELL, Raytheon Manufacturing Company, Waltham, Mass.

American Statistical Association.—"High-speed automatic computing machinery" was the discussion topic at a meeting of the local chapter of the American Statistical Association in Washington, D. C., on October 27, 1947, at 8:00 p.m. A survey of the national machine development program was presented by Dr. J. H. CURTISS, Chief of the National Applied Mathematics Laboratories, National Bureau of Standards. This was followed by a talk by Mr. J. L. McPHERSON of the Census Bureau on "The application of high-speed automatic digital computing machinery to statistical tabulation" (see DISCUSSIONS).

On the evening of November 10, 1947, the chapter presented talks on "The solution of statistical problems for automatic computing machines." In keeping with this topic, Dr. E. W. CANNON, Chief of the Machine Development Laboratory of the National Bureau of Standards, discussed "Instruction codes for high-speed automatic computing machines," and Mrs. IDA RHODES, also of the Bureau, in a talk on "Programming problems for solution," presented the coding sequence to be used on the new machines for several problems of a statistical nature.

National Electronics Conference.—On Tuesday, November 4, 1947, a meeting on electronic computers was held at Chicago, Illinois. Dr. J. W. MAUCHLY of the Electronic Control Company, Philadelphia, Pa., discussed computers from the block diagram, or functional, standpoint. This was followed by a talk by Mr. J. M. COOMBS of Engineering Research Associates, St. Paul, Minnesota, on the development of magnetic disc memory devices. Mr. O. H. SCHUCK, of the Minneapolis-Honeywell Company, also spoke on analogue potentiometer-type computers for aeronautical navigation. Most of the discussion at the meeting centered around the computing ability, reliability, and time required for construction of the proposed machines.

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z–III

1. LEE JOHNSON, "How to speed up slide-rule work," *Engineering News-Record*, v. 139, July 24, 1947, p. 112–114. 21 × 28.6 cm.

2. "Kinks and short-cuts—Winder production aided by slide-rule calipers," *Textile World*, v. 97, no. 10, Oct. 1897, p. 154–156. 21 × 28.6 cm.

"Loss of production time can be avoided in the winder room by using this slide-rule caliper for estimating the amount of yarn on unfinished cones at shift-changing time. Instead of allowing the machine to remain idle while a part of a set could be run, the operator can run the part set with assurance that piece-rate pay for the number of pounds run will be equitably apportioned. The calipers indicate the number of pounds run according to package diameter and spindle assignment."

3. R. G. MANLEY, "Computer for principal stresses," *Engineering*, v. 164, Oct. 10, 1947, p. 340–341. 26.4 × 36.1 cm.

An instrument consisting essentially of two horizontal slides, one vertical slide and three cursors, one of which carries a radius arm and protractor.

4. L. E. WADDINGTON, "A slide rule for the study of music and musical acoustics," Acoustical Soc. Amer., *Jn.*, v. 19, Sept. 1947, p. 878–885. 20 × 26.6 cm.

"Musicians are seldom concerned with the mathematical background of their art, but an understanding of the underlying physical principles of music can be helpful in the study of music and in the considerations of problems related to musical instrument design. Musical

data and numerical standards of the physics of music are readily adaptable to slide-rule presentation, since they involve relationships which are the same for any key. This rule adjusts relative vibration rates, degrees of scale, intervals, chord structures, scale indications, and transposition data, against a base of the piano keyboard. It employs and relates several standard systems of frequency level specification."

5. A. J. BARACKET, "Square-wave response," *Electronics*, v. 20, no. 8, Aug. 1947, p. 130. 21 × 28.5 cm.

"Nomograph correlates tilt of square wave after passage through uncompensated RC-coupled video or audio amplifier with low-frequency response of amplifier and time constant of coupling circuit."

6. IRA J. HOOKS, "Nomograph aids use of Boussinesq equation," *Civil Engineering*, v. 17, June 1947, p. 46–47. 21 × 28.6 cm.

Boussinesq's equation is $y = 3Pz^3/[2\pi(r^2 + z^2)^{5/2}]$.

7. R. E. LAFFERTY, "Voltmeter loading," *Electronics*, v. 20, no. 10, Nov. 1947, p. 132–133. 21 × 28.5 cm.

"Simple nomograph $[SE_L(E - E_H) = E_H(E - E_L)]$ gives true voltage in high-impedance circuit when measurements are made with two different voltage ranges of an ordinary low-sensitivity voltmeter. Underlying equations for voltmeter error are given, with examples of use."

8. KURT BENJAMIN, "Problems of multiple-punching with Hollerith machines," Amer. Statist. Assoc., *Jn.*, v. 42, Mar. 1947, p. 46–71.

9. BERTRAM J. BLACK & EDWARD B. OLDS, "A punched card method for presenting, analyzing, and comparing many series of statistics for areas," Amer. Statist. Assoc., *Jn.*, v. 41, 1946, p. 347–355.

10. PSYCHOMETRIC SOC. & IBM, *Proceedings of the Research Forum, Endicott, New York, August 26–30, 1946.* IBM, New York, 1947. 94 p., 21.6 × 28 cm.

CONTENTS: LEDYARD TUCKER, "Simplified punched card methods in factor analysis," p. 9–19; PAUL S. DWYER, "Simultaneous computation of correlation coefficients with missing variates," p. 20–27; ALBERT K. KURTZ, "Scoring rating scales after the responses are punched on IBM cards," p. 28–34; CHARLES I. MOSIER, "Machine methods of scaling by reciprocal averages," p. 35–39; W. G. COCHRAN, "Use of IBM equipment in an Investigation of the 'truncated normal' problem," p. 40–44; JOHN C. FLANAGAN, "Use of IBM equipment in obtaining stanines in the AAF," p. 45–51; JOHN V. McQUITTY, "Maximum use of mechanical aids in handling test results," p. 52–55; H. S. DYER, "Making test score data effective in admission and course placement of Harvard freshmen," p. 56–62; ERWIN K. TAYLOR, "The use of a single card column for recording variables with a range of 30 or fewer units," p. 63–67; HERBERT A. TOOPS, "The research possibilities of addends," p. 68–74; W. J. ECKERT, "Facilities of the Watson Scientific Computing Laboratory," p. 75–80; H. R. J. GROSCH, "Harmonic analysis by the use of progressive digiting," p. 81–84; WARREN G. FINDLEY, "The use of the IBM test-scoring machine in the N. Y. State scholarship

programs," p. 85–88; ARTHUR E. TRAXLER, "Accuracy of machine scoring of answer sheets marked with different degrees of excellence," p. 89–94.

11. A. D. BOOTH, "Two calculating machines for X-ray crystal structure analysis," *Jn. Appl. Physics*, v. 18, July, 1947, p. 664–666.

First two paragraphs: "Apart from large and expensive machines for performing automatically the whole range of crystallographic calculation, there remains a need for simple *ad hoc* devices to deal with particular aspects of the problem. If the current schemes for centralizing the latter stages of Fourier refinement come to fruition, this demand for the 'home made' and simple type of calculator is likely to be increased.

"It is the purpose of this paper to describe two such arrangements which the author designed and found useful in practical structure analysis. The first is especially valuable in the case of tetragonal space groups and found extensive application during the determination of the structure of pentaerythritol tetranitrate. The second is considerably simpler in principle and in design, and has been in constant and satisfactory use in structure analysis for the past four years." (See G. A. JEFFREY, R. Soc. London, *Proc.*, v. 183A, 1945, p. 388 and v. 188A, 1947, p. 222. Also *Math. Rev.*, v. 8, 1947, p. 606, S. H. C.)

12. GREAT BRITAIN, MINISTRY OF SUPPLY, Aeronautical Research Council, *Reports and Memoranda* no. 2144, Aug. 1944: R. A. FAIRTHORNE, *Mechanical Instruments for Solving Linear Simultaneous Equations.* London, His Majesty's Stationery Office, July 1947, 7 p. 23 × 30.4 cm. 1 shilling and 6 pence.

"*Summary:* The solution of linear simultaneous equations is necessary for the investigation of many important aeronautical problems, such as those of flutter, stability and vibration.

"Some mechanical instruments for the direct solution of these equations are reviewed, the principles used ranging from nomography to hydrostatics. It is concluded that, while almost any mechanical principle can be employed, the instrument can be successful only if carried out as a major engineering project.

"A practical limit to the value of any instrument is the time taken to set up the coefficients. A mechanical device would seem always to be inferior to an electrical in this respect.

"It is suggested that attention might be given to mechanical nomograms, such as Torres' Arithmophores, and, in the light of contemporary projection and motion picture technique, to extension of the principles of multiplane nomography."

For Arithmophores of LEONARDO TORRÈS QUEVEDO, see M. D'OCAGNE, *Le Calcul Simplifié*, third ed., Paris, 1928, p. 80–81, 158–167.

13. EGBERT HARBERT, *Vermessungskunde*. V. 1, third ed., Berlin, Verlag der deutschen Arbeitsfront, 1943. "Rechenhilfsmittel," p. 275–304.

The pages are pleasantly discursive. On p. 275–283 are discussed tables of NAPIER, BÜRGI, BRIGGS, F. G. GAUSS & GOBBIN, BREMIKER, VEGA, JORDAN, CRELLE, H. ZIMMERMANN and L. ZIMMERMANN; and then, p. 303–304, trigonometric tables of PETERS, F. G. GAUSS, BRANDENBURG, JORDAN, STEINBRENNER, BALZER & DETTWILER, LEUPIN. In all this, more space is devoted to various tables of Gauss than of any other author.

Slide rules, and in particular those of GUNTER, OUGHTRED, and SCHERER, are discussed, p. 283–289.

Pages 290–303 are devoted to Calculating machines with references to PASCAL, LEIBNIZ, HAHN, POLENI, THOMAS, ODHNER (always misspelled by Harbert), BRUNSVIGA, HAMANN, MERCEDES-EUKLID, and MILLIONAIRE.

14. W. MEYER ZUR CAPELLEN, *Mathematische Instrumente (Mathematik und ihre Anwendungen in Physik und Technik,* ed. by E. KAMKE & A. KRATZER, series B, v. 1). Second enlarged ed., Leipzig, Akademische Verlagsgesellschaft, 1944. x, 313 p., 250 figures and illustrations. Offset print, without correction of errors. Ann Arbor, Mich., Edwards Bros., 1947. Published and distributed in the public interest by authority of the Alien Property Custodian under License no. A1064. $10.50.

The present book of which the first edition appeared in 1941, describes a wide range of mathematical devices, mainly of German or continental origin. There is considerable emphasis on constructional detail and the actual use of the instruments.

The major part of the book is divided into five sections: Calculating Devices, p. 4–142; Geometrical Devices, p. 143–169; Differentiators, p. 170–178; Integrators and Harmonic Analyzers, p. 179–289. There is a bibliography, p. 290–305, listing 310 references, an excellent coverage of such material appearing during the past 20 years; 174 of these items were published 1936–1943.

The calculating device section begins with a description of various addition and multiplication devices. Specifically, the various differentials are described as well as similar triangle multipliers, the disc integrator as a multiplier, the three-dimensional cam, a WHEATSTONE Bridge multiplier and various mechanisms involving sliding rods, mounted in revolvable collars, which are used for multiplication, division and various trigonometric purposes. A device of this type for the solution of nomograms is described (p. 31–32).

The next portion of this section (p. 32–53) is mainly devoted to the slide rule. A slide rule of the "System Darmstadt" pattern is described in detail. The various scales and their uses are indicated and a table describing 89 formulae which can be almost immediately evaluated by means of the slide rule is given. There is also a description of certain more complicated logarithmic devices.

A detailed and exceedingly interesting description of desk type calculating machines of German manufacture follows (p. 53–131). The exterior of the machines is first described and the exact significance of the various operational cranks and auxiliary keys is given. Then the interior mechanisms are described with proper emphasis on the tens transmission. The verbal descriptions are clear but some of the photographs seem to have suffered from this offset reproduction and are too dark to be of effective assistance. The uses of these machines, and the usual, very useful, precisely stated procedures, are given in the final section of this chapter.

The following machines are described in detail: 1. A LEIBNIZ wheel machine made by "Rheinmetall," 2. Two-sprocket wheel (ODHNER) types, see *MTAC*, v. 2, p. 149, "Brunsviga" and "WALTHER," 3. A proportional lever type "Mercedes-Euclid," 4. The "Hamann," which has its own characteristic feed, the "Schaltklinken" (switching catch?) arrangement. There are briefer descriptions of the "Millionär," which splits the multiplication table, and of the "Continental" which uses a key stop or Comptometer principle.

The author seems to favor the handdriven machines and the smaller electrical devices and it may be that if a scientist uses a calculating machine only during a small part of his time, he may find these smaller machines just as effective for his purposes. However, among those who spend a good deal of time on calculations, there is certainly a demand for the maximum mechanization possible. For the desk machine, the dividing line seems to be at the automatic multiplication point. Machines with this additional ability cost more but in many cases the time saved, when computed in terms of salary, easily justifies the additional expense. And certainly in this country, the use of automatic punched card computing has made it possible to consider scientific questions which would be far beyond the capacity of desk type machines.

The calculating devices section ends with a chapter which considers devices for the solution of equations. The WILBUR and MALLOCK linear equation solvers (see *MTAC*, v. 1, p. 350; v. 2, p. 158) are described and also an electrical machine due to BODE, which uses reactances. There is an adjusting type of device due to RECK but the convergence question is

not considered. For a polynomial equation, the HART-TRAVIS machine (see *MTAC*, v. 1, p. 350) is described and also T. C. FRY's isograph (see *MTAC*, v. 1, p. 167), with which only the names DIETZOLD-MERCNER are associated by the author.

The geometrical devices part of the book contains chapters on various instruments for the accurate graphing of functions in cartesian or polar coordinates, changing scale and various graphical transformations as well as one on the instruments for drawing conic sections and the more general curves like the cycloids and spirals. Instruments for measuring arc lengths are also given here. In the next part on Differentiators, optical methods for determining tangents are described.

The Integrator part contains a description of the various planimeters, integraphs and integrometers. This part also contains a section on the differential analyzer. The precise procedure for the use of the PRYTZ planimeter is also given and also for certain related devices. The concluding part on the Harmonic Analyzers (p. 273–290) seems rather brief.

The reviewer feels that a German-English glossary of technical terms would be very helpful for reprints of this type.

The book is a valuable contribution to the literature. Its general plan is well conceived, and the various detailed descriptions and many of the photographs are excellent. It certainly should be available to those who own or operate one of the German machines, both to utilize the device with maximum effectiveness and in case of needed repair.

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15. S. VAJDA, "Shortcutting in multiplication on a calculating machine," *Math. Gazette*, v. 31, July 1947, p. 172–173.

16. HENRY G. WEISSENSTEIN, "Calculating machine furnishes shortcut method of computing P.I. of two lines," *Civil Engineering*, v. 17, Sept. 1947, p. 545. 21 × 28.6 cm.

"Computing the coordinates of the point of intersection of two lines is a rather tedious process if done the conventional way using the law of sines. With the help of a calculating machine considerable time can be saved using the method shown in this article."

NOTES

87. GERMAN ALTITUDE AND AZIMUTH TABLES.—DEUTSCHE SEEWARTE, (**a**) *Höhen und Azimute der Gestirne, deren Abweichung zwischen 30°S und 30°N liegt, für 50° Breite.* (**b**) *Höhen und Azimute der hellen Fixsterne bis zur dritte Grösse deren Abweichung grösser als 30°N ist, für 50° Breite.* Herausgegeben vom Reichs-Marine-Amt. Berlin, 1916, (**a**) xxiii, 377 p., (**b**) xii, 88 p. 20.5 × 30 cm.

(**a**) This volume is clearly one of a series, but neither the introduction nor preface gives any indication of the extent of that series. The preface of the particular volume under review is dated July, 1916, and that of the exactly similar volume for latitude 70°, September, 1917; there also exists a volume for latitude 55°, preface dated June 1916, which is printed by a photo-lithographic process from manuscript figures. Although the ms. is very good it cannot compare in legibility with printed figures; surprisingly the format is smaller than for the printed volumes, the overall size being 20.5 × 27.5 cm. The preface hints that the manuscript has been reproduced by photography to lessen the chance of errors occurring in the process of letterpress printing; presumably, experience rapidly led to placing legibility higher than freedom from error! In all the volumes users are begged to communicate errors to the compilers.

The main tables comprise the most extensive tables of altitude and azimuth for a given

latitude known to the writers; in the field of solutions of spherical triangles they clearly take a major place and deserve a detailed description. Altitudes are given to 0'.1 and azimuths to 0°.1—the usual precision required for marine navigation—but the unusual, and important, feature is that the interval in declination is 10' and that in hour angle 1ᵐ of time or 15' of arc.

The tables are divided into two parts, the first (and larger) referring to declinations of the "same" name as the latitude and the second to declinations of "opposite" name. The principal argument is hour angle in time, and each part is divided into sections of one hour, easily found by a large marginal index, containing tabulations for declinations 0(10')29°50'. Each pair of facing pages contains the tabulations for two degrees of declination. The left-hand page is concerned exclusively with altitude; with vertical argument 0ᵐ(1ᵐ)59ᵐ—note that the hourly values are not repeated—the altitude is tabulated for twelve values of the declination; no differences or variations are given. On the right of the right-hand page, the azimuth is tabulated for each minute of hour angle and for each half-degree of declination. Whereas for the altitude, there is no repetition, the azimuth is repeated for each even degree of declination; why this is done is by no means clear.

The really interesting feature is the double-entry interpolation tables for the altitude; these are tabulated on the left of the right-hand page and give directly the increment to be applied to the tabular altitude, for each minute of declination (horizontal argument) and for some convenient interval (usually 6ˢ) of hour angle. Generally one such table, based on mean values of the variations, is given for each two degrees of declination and each 10ᵐ of hour angle; near the meridian, when the variation of the difference for 1ᵐ is large but the difference itself is small, tables are given more frequently using a larger interval of 10ˢ, 20ˢ, 30ˢ or even 60ˢ. In the volume for latitude 55°, there is little regularity in the interpolation tables; intervals of 20ˢ, 10ˢ, 5ˢ, 12ˢ, 6ˢ, 4ˢ, 3ˢ are used without apparent plan, the 60ˢ value always being included. The later volumes indicate much more care in the arrangement, the 60ˢ value being omitted to allow of one interpolation table to correspond to 10ᵐ of the main table.

Now the maximum variation of altitude for 1' of declination is 1'.0, on the meridian, and for 6ˢ of hour angle is 1'.5 cos(latitude) on the prime vertical, which for these high latitudes never exceeds 1'.0. A value of the altitude formed by the addition of two direct entry quantities—one from the main table, and one from the interpolation table—will thus never be in error by much more than 0'.8. But this maximum error is erroneously given as 0'.4 in the volume for latitude 55° and 0'.5 in the others; perhaps this assumes that the declination is known only to the nearest minute. In any case, the erroneous statement is made that altitudes correct to 0'.1 can be obtained by mental interpolation in the inter-polation tables; this ignores several factors, chief among them being the fact that mean differences are used. Errors up to at least 0'.3 can arise from this cause, especially near the meridian.

Nevertheless, the provision of double entry interpolation tables of this form is an excel-lent principle and one that should be developed for tables of this nature.

In the case of the azimuth, a small table of nine entries, for 0', 10', 20' of declination and for 0ˢ, 20ˢ, 40ˢ of hour angle, suffices to give corrections to the nearest 0°.1.

Generally, tabulations are given for all altitudes of 2° or greater; the squaring off of pages or half pages involves giving negative altitudes and these are given as low as 3° below the horizon. A thick zigzag rule separates positive from negative values, and signs are also given on all pages where negative values occur.

Fundamental values of the altitude to 0'.01 were computed by the Deutsche Seewarte, using seven-figure logarithms, for every 4ᵐ of hour angle and every degree of declination; these values were then subtabulated, using second differences. The statement that the tabu-lar values of the altitude are correct to 0'.05 is thus optimistic. Comparison with H.O.214 shows that about half the comparable entries differ in the end-figure, and independent com-putation confirms the accuracy of the H.O.214 figures in spite of their known unreliability (see, for instance, C. H. SMILEY's analysis in MTE 93). A casual examination has, however, failed to find many gross errors, though about one value in every ten pages has already been

corrected in ink in the particular copy under review, and one column of values is wrongly printed.

The general arrangement of the tables is adequate, though it could undoubtedly have been improved by combining the tabulations for altitude and azimuth. The lay-out is far from perfect, the chief criticism being the use of characterless modern face figures, which though large are very difficult to distinguish from each other. The use of rules instead of spaces for horizontal divisions further adds to the crowded appearance of the page. It would be easy to get 60 lines of smaller and well spaced type in the 8½ inches used here, with far greater legibility. In other respects the books are well printed and bound.

Several auxiliary tables are given, comprising the usual corrections to observed altitude for refraction, dip, semidiameter and parallax, as well as mean places of stars with declinations between ±30°. It is noted that the Sun's semi-diameter is assumed constant at 16′.0.

H.O.214 takes 24 pages for each degree of latitude; the present volume takes 377 pages. Tabulation at a small interval of latitude would clearly be impossible. The explanation lies in the fact that the tables are designed for use with long intercepts plotted on a stereographic chart; on such a chart all circles (great or small) on the Earth are circles. The centre of the projection is used as an assumed position. The position line determined by an observation of altitude will be tangential to the straight line drawn perpendicular to the intercept (normally regarded as the position line itself) and concave to the sub-stellar point; the departure from the straight line is a simple function of the radius of the circle, i.e. of altitude, and distance from the intercept, which is known to sufficient accuracy by the D.R. position. An approximation to the position line, in the vicinity of the D.R. position, will then be the tangent to the position line at that point; this is constructed by simple geometry on the chart. Each volume contains the table giving the deflection from the straight line—from which table, in fact, it would be easy to draw the circular arc. Specimen charts are shown to be available for latitudes N.45°, N.50°, N.55°, and N.70° and for multiples of 7½° in longitude. Each chart caters for an area roughly 11° square, so it would seem unnecessary to have them, or the volumes of tables, more frequently than every 10° in latitude. All the examples work from the centre of one or other of these charts and, accordingly, interpolate for hour angle; the scale of the charts is small (1′ = 1 mm. or ½ mm.), and the examples use 1′ as the working unit for everything except the calculated altitude from the tables. The most amazing feature of these elaborate tables is that the tabular accuracy of 0′.1 is not used; carefully designed and used tables to the nearest minute would give adequate accuracy with the declination and observed altitude to the nearest minute only. It would also have been possible to use a stereographic graticule independent of longitude and thus to avoid interpolation for hour angle; but this would involve transfer of the position line to another working chart.

As navigational tables, these can hardly have been very successful; as solutions of a spherical triangle they quite definitely have an interest.

(b) This volume is a companion volume to (a); a similar volume (20.5 × 27.5 cm.) for latitude 55° exists and is photographically reproduced from manuscript. No knowledge is available of other volumes.

Altitudes, to 0′.1, and azimuths, to 0°.1, are tabulated for 22 stars (including *Polaris*), with declinations greater than N.30°, for every minute of hour angle while above the horizon and not within 20° of the zenith; considerations of pagination give rise to altitude greater than 70° and to negative altitudes.

Each star is allotted four pages, each page containing 3 hours of hour angle. For each minute of each hour are tabulated the altitude and azimuth corresponding to the mean declination for 1917.5; for each 10^m of hour angle an interpolation table, based on mean differences, gives the increments to be applied to altitude and azimuth for multiples of 6^s of hour angle. At the foot of each column (i.e. for each hour) are given, for 1922.5, 1927.5 and 1932.5, the corrections to be applied to altitude and azimuth for precession in declination.

The tables are printed in the same style as the main volume and are intended for use in the same manner. The values of the altitude were computed for every four minutes using

six-figure logarithm tables and six-figure natural sines; the azimuths were interpolated (where possible) from "den amerikanischen Azimuttafeln für jede zehnte Minute des Stundenwinkels" (H.O.120), but otherwise calculated.

The most remarkable feature of the tables is the use of $0'.1$ as unit of altitude and thereafter the discarding (in the examples) of the advantages of this accuracy; this is borne out by the sketchy method of applying corrections for precession. It would have been simple to expand the correction table to give complete corrections with argument declination of star.

·J. B. PARKER & D. H. SADLER

88. HARRY BATEMAN (29 May 1882–21 January 1946).—Harry was a table-maker and a most learned and inspiring guide in the tabular field. Our brief obituary notice was in *MTAC*, v. 2, p. 77 (correction p. 375). This was followed by a tribute from E. T. BELL in *Quarterly of Applied Math.*, v. 4, 1946, p. 105–106, and a "List of Publications of Harry Bateman" by his adopted daughter JOAN MARGARET, p. 106–111, 202 titles, 1902–1946. Eight of these titles refer to problems originally published in the *Educational Times*, London, but the references given are not to this periodical but to volumes of the second series of *Mathematical Questions and Solutions from 'The Educational Times,'* often referred to by *E.T.R.* (Educ. Times Reprint). Such an error is excusable in this case, but it is hardly to be condoned when copied by the contributor of the memoir for the records of the R. Soc. London. Since there are many omissions in Miss Bateman's list (and the other list) of such problems, it may be well to put them on record. First of all, for the following 7 problems in the *Educ. Times* no solutions were ever given; hence these did not appear in *E.T.R.*: 14923 (1901), 15057 and 15255 (1902), 15550 and 15558 (1904), 16239 (1907), and 17104 (1911). Then there were the following items in *E.T.R.*, s. 2, 1902–1912, which Miss B. does not list: 10728, v. 1, p. 108, sol.; 14873, v. 1, p. 86, sol.; 14900, v.1, p. 117, prop., sol.; 15042, v. 2, p. 77, prop.; 14961, v. 3, p. 62, prop.; 15097, v. 3, p. 28, prop.; 15158, v. 3, p. 120, prop.; 15182, v. 3, p. 108, sol.; 15184, v. 4, p. 38, prop., sol.; 15294, v. 4, p. 94 and v. 17, p. 82, prop.; 15418, v. 6, p. 56, sol.; 15440, v. 6, p. 68, prop. (Miss B. and Dr. Erdélyi incorrectly refer to v. 5 for this); 9388, v. 7, p. 17–18, sol.; 15896, v. 10, p. 66–67, prop., sol.; 15997, v. 11, p. 53–54, prop., note, and v. 15, p. 72, prop.; 16112, v. 12, p. 34, prop., note; 16090, v. 12, p. 97, prop., sol.; 16304, v. 14, p. 48, prop.; 15997, v. 15, p. 72, prop.; 16215, v. 19, p. 51, prop.; 15388, v. 19, p. 54, prop.; 16979, v. 20, p. 34–35, prop., sol., p. 79, prop.; 17119, v. 22, p. 66–67, prop., sol.

In London Math. Soc., *Jn.*, Oct. 1946, issued Aug. 1947, v. 21, p. 300–310, is a memoir by A. ERDÉLYI. A more elaborate memoir—by far the best to the date of its publication—by Erdélyi, with a bibliography of 196 titles (8 *E.T.R.* titles being combined together as no. 1), and a splendid portrait, are given in R. Soc. London, *Obituary Notices of Fellows*, v. 5, Feb. 1947, p. 591–618. Apart from slips and omissions to which we have referred above, our misprint of the name STEINITZ is also copied. There is a brief tribute to Bateman in Academia Nacional de Ciencias Exactas, Físicas y Naturales de Lima, Peru, *Actas*, v. 10, fasc. 1–2, 1947, p. 99. He was elected a Corresponding Member of this Academia, Dec. 28, 1943. In Amer. Math. Soc., *Bull.*, v. 54, Jan. 1948, p. 88–103, Professor F. D. MURNAGHAN, of the Johns Hopkins University, has an interesting memoir, accompanied by a bibli-

ography substantially that of Miss Bateman, with all of its errors. Except for a portrait Professor Murnaghan has informed me that this is not very different from what he is to contribute to the National Academy of Sciences, *Biographical Memoirs.*

R. C. A.

89. THE MAGNITUDE OF HIGHER TERMS OF THE LUCASIAN SEQUENCE 4, 14, 194, $\cdots$.—The successive members of this series are connected by the defining relation $s_k = s^2_{k-1} - 2$. The numbers of digits involved respectively in each of the first ten values of s_k are 1, 2, 3, 5, 10, 19, 37, 74, 147 and 293. While using this sequence in testing the prime or composite character of certain numbers of the form $2^p - 1$, where p is an odd prime, the writer became more and more impressed with the rapid growth in size of s_k as k increases, until finally he yielded to the temptation of calculating the leading figures and more especially the total number of digits in s_{226} and in s_{388}. The first of these two terms was chosen because M_{227} had been investigated by the author, and the second because of its close approach to the power of 2 for which 2^{p-11} as multiplier would move the decimal point in log s_{10} more than 120 places to the right, that is, beyond the computed approximation to log s_{10} (*vide infra*). [For $p \leq 257$, $2^p - 1 = M_p$ is called a Mersenne number.[1] Regardless of the limit $p = 257$, M_p is prime when, and only when, $s_{p-1} \equiv 0 \pmod{M_p}$. 227 and 389 are prime.] The details of this investigation will now be presented because they may satisfy the scientific curiosity of other enthusiasts in the same field.

In order to diminish the influence of the subtractive 2 in $s_k = s^2_{k-1} - 2$ it was convenient to commence with the value of $s_{10}(=a)$. Then the leading terms of the expansion of s_k are found to be

$$(1) \qquad s_k = a^b\{1 - (2^{k-10})/a^2 + \cdots\}, b = 2^{k-10}.$$

It is easy to show that the second term between the braces plays no part in the subsequent calculations because of the inferior degree of approximation set by the arbitrarily chosen limit of 120 decimal places in log a.

In order to compute a^c, $c = 2^{216}$, and a^d, $d = 2^{378}$, it was necessary to find ln a and convenient to convert to log a. Of the 293 figures in a only the following were used, namely 68 72968 24066 44277 23883 74862 31747 53092 42471 54108 64667 17521 92618 58308 84874 05790 95796 47328 83069 10256 10434 36779 66393 55951 72042. The natural logarithm of this number was obtained by applying the method of initial factoring followed by the addition of negative radix logarithms as explained in detail in the author's book entitled: *Original Tables to 137 Decimal Places of Natural Logarithms of the Form $1 \pm n \cdot 10^{-p}$, $\cdots$*. There were 53 factors of this form in addition to the preparatory factors 10^{-292} and $\frac{1}{8}$. The final factor "$1 \pm c \cdot 10^{-q}$" had 61 zeros following 1 and preceding $+19600$ 96931 31584 47643 10463 95099 12432 46152 37049 29318 80748 28301. Finally ln a = ln s_{10} equalled 674.28244 32255 06154 81602 37298 21679 84334 18147 69391 36843 76414 57795 54327 56149 33219 31575 00001 34886 45211 31809 10303 52158 97494 11743, the increase in the total number of figures presented being due to the addition of $292 \ln 10$. Multiplication by the modulus (log e) then gave log s_{10} = 292.83714 43370 80010 66463 76818 65779 91989 91616 19717 85767 99035 37433 53371 72561 53630 91121 52243 44309 03398 12340 09801 71356 22384 20013.

The first one of the desired results was obtained by multiplying the last number by 2^{216}, that is by 1 05312 29166 85571 86697 91802 76836 70432 31889 50954 00549 11125 43109 77536. Hence $c \log s_{10} = \log s_{226} = 308$ 39350 75581 39495 52047 79655 72677 97216 29131 52823 21013 64722 49007 38462.82923 38824 55508 95824 25948 08561 34247 34163 42090 19894 95461 $\cdots$. The preceding characteristic (increased by one) shows clearly the enormous value of the 226th term of the Lucasian sequence 4, 14, 194, $\cdots$. For present purposes advantage will not be taken of the knowledge of all of the first 55 digits of the mantissa. Suffice it to conclude that $s_{226} = (10^{0.82923388\ldots}) \times 10^{308\cdots462} = (6.74891\cdots) \times 10^{308\cdots462}$ and that very roughly the number of figures in s_{226} is 3×10^{67}.

Similarly for the case of s_{388}. The multiplier of $\log s_{10}$ was 2^{378} in its expanded form, that is 6156 56346 81866 37376 91860 00156 47439 65704 37092 61010 22604 18669 20844 41339 40267 96439 15803 34791 02325 76806 88760 35623 48544. Then $\log s_{388} = 2^{378} \log s_{10} = 18 \ 02870 \ 46495 \ 37642 \ 27934$ 36482 96937 59970 26529 12244 19199 46086 61354 33080 80671 26804 26804 16844 29101 89242 85304 79752 21848 58395.65941 3. Therefore $s_{388} = (10^{0.659413\ldots}) \times 10^{180\cdots395} = (4.56470\cdots) \times 10^{180\cdots395}$. For s_{388} the number of figures is roughly 1.8×10^{116}.

Obviously $\log s_{10}$ may be multiplied by 2^{p-11} in order to obtain two or more leading figures of s_{p-1} which correspond to the 74 prime numbers beginning with 13 and ending with 401.

If the value of s_{388} were written out in full as my ruled paper regulates, the length of the strip would be about

$$4.6512776 \times 10^{(1.80287\ldots)10^{116}-19} \text{ parsecs.}$$

A professional astronomer has recently assured me that the paltry number 2 billion parsecs can be set safely as the present superior limit of observable celestial objects.

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[1] See *MTAC*, v. 1, p. 333, 404; v. 2, p. 94, 341.

90. Mathematical Table Makers.—*Mathematical Table Makers. Portraits, Paintings, Busts, Monuments, Bio-Bibliographical Notes.* By R. C. Archibald. (*The Scripta Mathematica Studies*, no. 3.) New York 33, Yeshiva University, Amsterdam Ave. and 186th St., 1948, vi, 82 p., 20 plates. 16.6 × 24.7 cm. Cloth $2.00.

This little book is a thorough revision, rearrangement, and considerable enlargement, with new illustrations, of two articles which appeared in *Scripta Mathematica* in 1946. The following 53 Table Makers are considered (a star indicating an accompanying portrait): *Airey, Anding, Babbage, Bauschinger, Becker, Bessel, Bierens de Haan, Borda, Brown (E. W.), Bürgi, Burrau, Cohn, *Comrie, *Cunningham, Dase, *Davis, *Dickson, *Dwight, Glaisher (J.), *Glaisher (J. W. L.), Hoëne-Wroński, Hoppe, Hutton, Jacobi, *Kepler, *Kraïtchik, LaLande, *Legendre, *Lehmer (D. H.), *Lehmer (D. N.), Lodge, Lohse, Lommel, *Lowan, Markov, Martin, *Miller, *Napier, Nielsen, *Pearson, Peirce, *Peters, Rivard, Sang, Sharp, Sheppard, Stevin, Stieltjes, *Tallqvist, *Thompson, Turner, *Uhler, Viète. For each individual there are biographical notes followed by information under three headings: P (sources for a portrait,

painting, etc.) or **PB** (biographical material containing **P**); **B** (references for biographical data); and **T** (list of the individual's tables).

The portraits are distributed in the book as 5 groups of four plates. The portrait of Peters is with Comrie and the astronomer Kruse. The most extensive discussion of **P** is that for Kepler (5 p.), so many of whose alleged published portraits and busts are not of Kepler at all. Of Borda, Bürgi (2), Kepler (2) and Stevin public monuments are listed, as also are medals struck in honor of Kepler and Stevin.

Under **T** the most extensive title list (53) is for Cunningham; Airey (49) and J. W. L. Glaisher (48) come next.

According to countries of birth for the Table Makers the distribution is as follows: Australia (1), Belgium (1), Denmark (2), England (12), Finland (1), France (5), Germany (12), Holland (2), India (1), New Zealand (1), Poland (1), Roumania (1), Russia (2), Scotland (2), Switzerland (1), U. S. A. (8).

By his will Cunningham left to the London Mathematical Society: (i) £1000 for the improvement of the method of factorization of large numbers; (ii) £2000 for the publication of his unpublished mathematical works, and the completion and publication of his mathematical manuscripts; (iii) his library of mathematical books. Of the residue of his estate he left one-twelfth [about £3000] to the London Mathematical Society, and one-twelfth to the British Association, Mathematical Subsection, for preparing new mathematical tables in the theory of numbers. (*Times*, London, May 12, 1928, p. 10; BAAS, *Report 1930*, p. 251–252.)

The tables of D. N. LEHMER listed in Q26 were inadvertently omitted.

91. PORTRAITS OF TABLE MAKERS.—In *Nature*, v. 160, 22 Nov. 1947, p. 721–722, is a report by JAMES T. KENDALL of an "International Congress for Technical Education" held at Darmstadt, Germany, July 31–Aug. 9. Among the 50 foreign visitors were 15 from Great Britain, and among 200 papers read was one by L. J. COMRIE on "Calculating machines and mathematical tables." These papers are later to be published by the Technische Hochschule in Darmstadt. In illustrated reports of the Congress in *Darmstädter Echo*, 2 August 1947, is a reproduction of a photograph of Prof. FRITZ EMDE (b. 13 July 1873) and Dr. Comrie, seated at a table in a restaurant. In a letter dated 26 Aug. 1947, to Dr. Comrie, Prof. Emde stated (1) that Teubner had just received publishing license from the Russians for reprinting the fourth edition of the Jahnke & Emde work, and for a new edition of Emde's Elementary Functions destroyed by air raids in 1944; and (2) that the Russians had ordered resumption of the former bimonthly, *Zeitschrift für angewandte Mathematik und Mechanik*, as a monthly publication.

In *Simon Stevin, Wis- en Natuurkundig Tijdschrift*, v. 25, no. 1, 1946, there is a large folding frontispiece of *Simon Stevin*, a reproduction of a painting, by an unknown artist, in the University Library, Leyden. This periodical is a new one replacing three earlier publications: (a) *Christiaan Huygens* (v. 1–24, 1921–1946), (b) *Wis- en Natuurkundig Tijdschrift* (v. 1–12, 1921–1945, perhaps later numbers); (c) *Mathematica B, Tijdschrift voor allen die de Hoogere Wiskunde beoefenen*, v. 1–13 (1932–1946). This new journal is not the first one so named; more than 40 years earlier appeared *Le "Simon Stevin," Journal des Candidats aux Écoles Spéciales*, Brussels, edited by J. Stevens, v. 1–2, 1905–06, paged continuously, 1–400; a mimeograph print with printed covers.

There are portraits of PAFNUTIĬ L'VOVICH CHEBYSHEV (1821–1894) in *Polnoe Sobranie Sochineniĭ P. L. Chebysheva*, v. 1, 1944; and in *Oeuvres de P. L. Tchebychef*, 2 v. 1899–1907, 3 portraits; see *MTAC*, v. 1, p. 440.

There is a portrait of MIKHAIL BORISOVICH OSTROGRADSKIĬ (1801–1861) in *Les Mathématiques dans les Publications de l'Académie des Sciences* 1728–1935. *Répertoire Bibliographique*. Moscow, 1936, p. 64; see *MTAC*, v. 1, p. 440; v. 2, p. 252–253.

Of J. H. Lambert there are portraits in his: (a) *Schriften zur Perspektive, herausg. u. eingeleitet* von MAX STECK, Berlin, 1943, Plate I (portrait), Lambert Monument in Mülhausen, Alsace (Plate III), Portrait relief on Monument (Plate IV); and (b) *Mathematische Werke*, v. 1, 1946 (see *MTAC*, v. 2, p. 339).

There are portraits of W. J. ECKERT, director of the department of pure science of the Watson Computing Laboratory, on p. 12 of *IBM Selective Sequence Electronic Calculator*, New York, 1948, and in *Business Machines*, v. 30, March 15, 1948, p, 3. Under his direction tables have been computed.

R. C. A.

92. PORTUGUESE NAVIGATION TABLE.—The table in question is ABEL FONTOURA DA COSTA & FRANCISCO PENTEADO, *Tábuas de Altura e Azimute. Comemorando o* 70° *aniversário do Club Militar Naval.* (Supplement to *Anais do Club Militar Naval*, November 1936.) Lisbon, Imprensa da Armada, 1936, v, 21 p. 13.2 × 24.3 cm.

These tables are comparable in size and content to those of AGETON, *Dead Reckoning Altitude and Azimuth Table* (H.O.211, RMT 104). The astronomical triangle is divided into two right triangles by a perpendicular from the celestial object upon the meridian; the author acknowledges that his formulae are patterned after Ageton's, "com uma pequena modificação na decomposição do triângulo de posição, de que resultam regras mais simples a aplicar." Actually the changes in the formulae result from calling the length of the perpendicular from the celestial body upon the meridian $90° - \psi$ instead of R, and the declination of the foot of the perpendicular $90° - \gamma$ instead of K. The rules for the use of the tables appear to be essentially equivalent to those advocated by Ageton. It is hard to see how one can say that they are simpler.

The number of pages has been cut in half since tabular values of $10^5 \log \csc x$ and $10^5 \log \sec x$ (called here C and S; A and B by Ageton) are given for each integral minute of arc rather than each half minute as in Ageton. It is the values of C which are given in heavy type; this is in contrast to Ageton's use of heavy type for B (or S). Values of C and S less than 665 are given to one decimal; this marks an improvement over Ageton's limit of 239.

No warning is given of the difficulties encountered when γ is near zero, Ageton's K near 90°. The larger interval of the argument adopted here makes this more serious.

Upon comparing 600 values of C with the corresponding values of A in Ageton, 34 places were found where the values differed by one unit in the last place. Using VEGA's 7-figure *Logarithmisch-trigonometrisches Handbuch*, FONTOURA & PENTEADO were shown to be correct in 30 of the cases, and the other four were indecisive. Upon examining these four cases again with an 8-figure table, it was found that Fontoura & Penteado were correct in all four cases. This would seem to indicate that the tabular values are definitely superior to those given in Ageton.

The usual auxiliary tables of refraction, dip of horizon, and parallax corrections are provided. The printing and paper are very good and an excellent thumb index is provided.

CHARLES H. SMILEY

EDITORIAL NOTE: There are two errors on p. 19: 86°10′, S, *for* 111487, *read* 117487; 88°25′, S, *for* 156861, *read* 155861.

QUERIES

26. SOME CLOTHOID OR EULER SPIRAL TABLES.—The curve is defined by the equations $x = \int_0^s \sin (\tfrac{1}{2}t^2/a^2)dt$, $y = \int_0^s \cos (\tfrac{1}{2}t^2/a^2)dt$, and with asymptotic points at $(\tfrac{1}{2}a\pi^{\frac{1}{2}}, \tfrac{1}{2}a\pi^{\frac{1}{2}})$, $(-\tfrac{1}{2}a\pi^{\frac{1}{2}}, -\tfrac{1}{2}a\pi^{\frac{1}{2}})$. The intrinsic equation of the curve $Rs = a^2$, shows that the radius of curvature of any point of the curve is inversely proportional to the length of arc to that point. All of these results were found by Euler (1744, 1781) who imagined half of the curve with its infinite number of whorls, and hence the name Clothoid given by Cesàro (1886). Euler studied the curve in connection with the solution of a problem of an elastic spring. The similar problem of an elastic lamina was considered earlier (1694) by JACQUES BERNOULLI, but there is no indication that he had any conception of the real form of the curve. These integrals became of importance in optics after Fresnel's diffraction study (1818). Since Cornu plotted the curve accurately (1874) the spiral is sometimes called Cornu's spiral. I gave a summary of history and bibliography of the curve in *Amer. Math. Mo.*, v. 25, 1918, p. 276–282 (the name "Peters" here should be replaced by "Gilbert"). One of the first published papers of D. N. LEHMER was "Cornu's spiral as a transition curve," *California Jn. Technology*, v. 3, 1904, p. 71–82. There are here seven tables to enable the engineer to lay out the curve from a point with curvature zero up to that of an appropriate circular arc, then back through a Clothoid arc to another straight line. Among many engineering treatments of the Clothoid see A. L. HIGGINS, *The Transition Spiral and its Introduction to Railway Curves*, London, 1921; there are various tables p. 100–107.

If $a^{-2} = \pi$, the equation of the Clothoid may be written

$$x = S(u) = \tfrac{1}{2} \int_0^z J_{\frac{1}{2}}(t)dt, \qquad y = C(u) = \tfrac{1}{2} \int_0^z J_{-\frac{1}{2}}(t)dt$$

that is, Fresnel integrals, of which there are many tables.

In a book published in 1943 by the Herbert Wichmann-Verlag, Berlin-Grunewald I find the following book listed: SCHÜRBA, *Klothoiden-Absteckta-feln. Anleitung zu Entwurf, Berechnung und Absteckung*, 143 p., "grossoktav-format," 47 figures. Elsewhere the date of publication is given as 1942. Who was Schürba? Where may a copy of this book be consulted? Are there other books of this kind?

R. C. A.

QUERIES—REPLIES

34. AN ENGEL TABLE (Q6, v. 1, p. 131, QR7, v. 1, p. 170–171).—Some information has been already recorded regarding the following table of ERNST ENGEL: *Zehnstellige Tafeln der Sinus, Cosinus und Tangenten für die dezimale Teilung des Nonagesimalgrades. Generaldirektion des Grundsteuerkatasters (Österr. Triangulierungs- und Kalkulbureau)*. Vienna, Bundesvermessungsamt, 1920, 95 p. 19.6 × 30 cm. Brown University has recently acquired a copy of this work, which seems to be a fact worth noting, since there does not appear to be any other copy in the United States; nor do I know of any copy in England.

R. C. A.

CORRIGENDA

V. 1, p. 118, l. 6, *for* 1906, *read* 1904; l. 7, *for* 1908, *read* 1905; l. 9, *for* 1908, *read* 1906;
p. 122, l. −8, *for* 1908, *read* 1906; v. 2, p. 139, l. 11, *for* v. 33, 1906, p. 9, *read* v.
33, no. 9, 1906, p. 12; p. 211, l. 3, *for* $5 \cdot 10^5$, *read* $6 \cdot 10^6$; l. 16, *for* x, *read* X; p. 291,
l. 22, *for* Sonine, *read* Sonin; v. 3, p. 13, l. −9 and p. 14, l. −9, *for* inverse, *read*
transpose [the author's ms. was correct—the change was due to an editorial lapse];
p. 32, l. −16, *for* DAVIES, *read* DAVIS; p. 37, l. 5, *for* P_{20} [twice], *read* P_{26}; p. 40, l. 8,
for BOURQUART, *read* BOURQUARD; p. 42, l. 19, *for* Steltjes', *read* Stieltjes'; p. 58, l. 4,
delete at Princeton University; p. 62, l. 20, *delete* the late.